Linear Time Runs over General Ordered Alphabets

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Maximal Periodic Substrings

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Compute All Runs	$\boldsymbol{\Theta(n)}$		
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	Kolpakov & Kucherov '99	**this work '21**	**e.g. Main & Lorentz '94**

- A Note on Alphabet Types
- **Reduction of Runs to Next Smaller Suffixes and LCEs**
- Linear Time Next Smaller Suffixes
- Linear Time LCEs
- Practical Aspects & Conclusion

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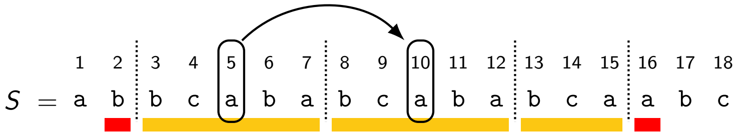

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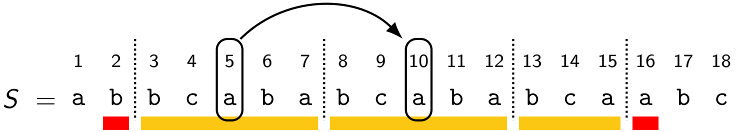

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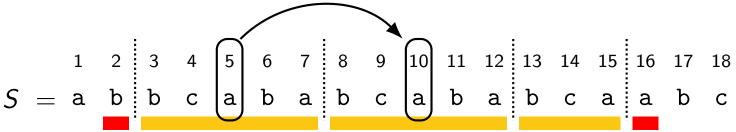

S_5 = a b a b c a b ...

S_6 = **b** ...

S_7 = a b **c** ...

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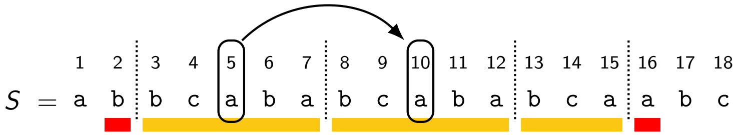

$$S_5 = \texttt{a b a b c a b} \ldots$$
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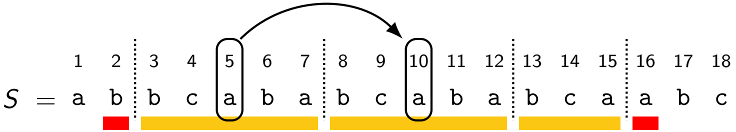

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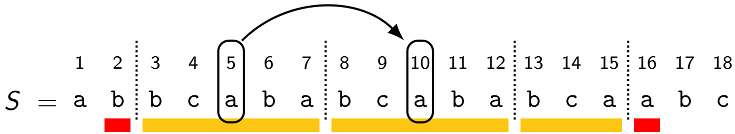

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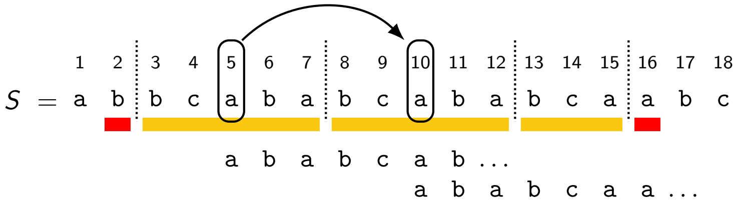

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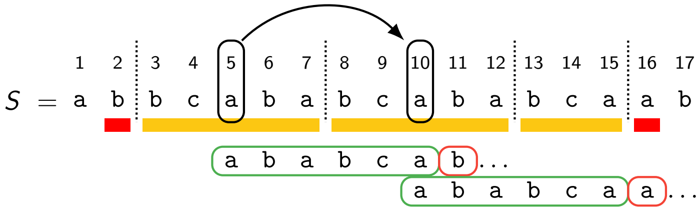

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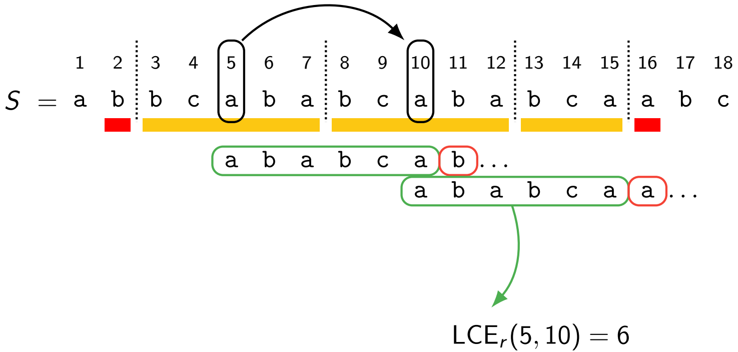

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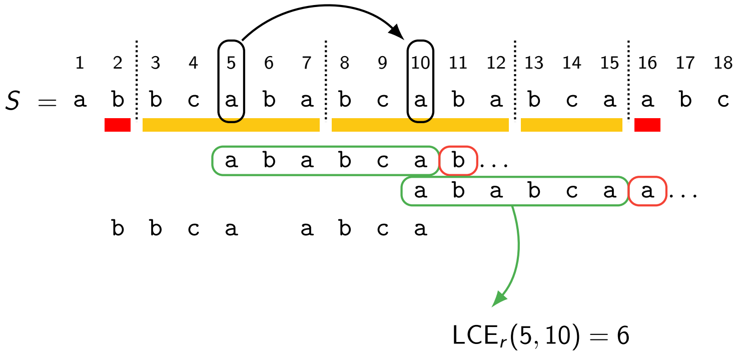

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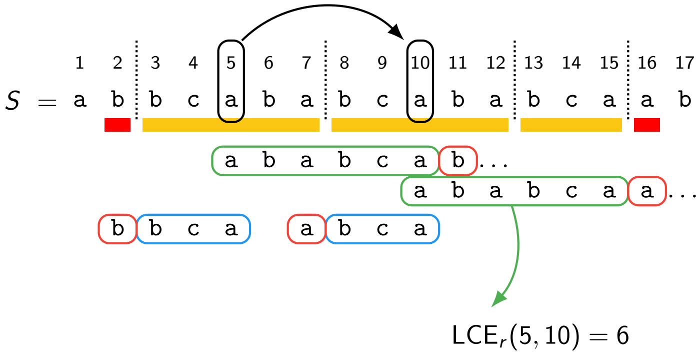

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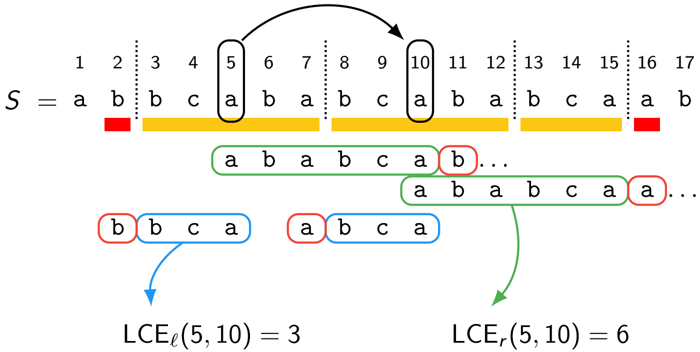

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    if $e - s + 1 \geq 2p$ **then**
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9: $\quad\quad\quad S[s..e]$ is a run with period p.
10: Repeat lines 2–9 with next-larger-suffix edges.

$\mathcal{O}(n \lg^{2/3} n)$ [Kosolobov 2016]

$\mathcal{O}(n \lg \lg n)$ [Gawrychowski et al. 2016]

$\mathcal{O}(n\alpha(n))$ [Crochemore et al. 2016]

- A Note on Alphabet Types
- Reduction of Runs to Next Smaller Suffixes and LCEs
- **Linear Time Next Smaller Suffixes**
- Linear Time LCEs
- Practical Aspects & Conclusion

Computing Next Smaller Suffixes

Computing Next Smaller Suffixes Values

Computing Next Smaller ~~Suffixes~~ Values

Require: Array $A[1..n]$
Ensure: Arrays nsv and psv

1	2	3	4	5	6	7	8	9	10	11	12	13
1	6	3	8	12	5	11	2	7	4	9	13	10

```
for i = 1 to n do
    j ← i − 1
    while j > 0 ∧ A[j] > A[i] do
        nsv[j] ← i
        j ← psv[j]
    psv[i] ← j
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1	2	3	4	5	6	7	8	9	10	11	12	13
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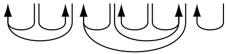

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for $i = 1$ **to** n **do**
  $j \leftarrow i - 1$
  while $j > 0 \wedge A[j] > A[i]$ **do**
    $\text{nsv}[j] \leftarrow i$
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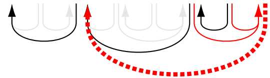

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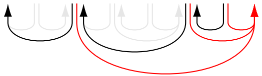

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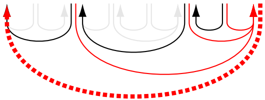

Computing Next Smaller ~~Suffixes~~ Values

Require: Array $A[1..n]$
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for $i = 1$ **to** n **do**
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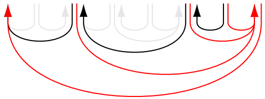

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Require: Array $A[1..n]$
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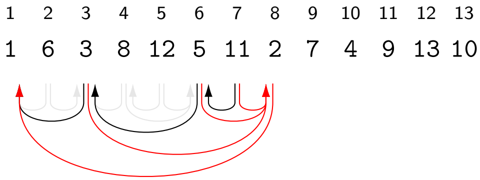

- after every iteration of line 3, we assign either nsv$[j]$ or psv$[i]$

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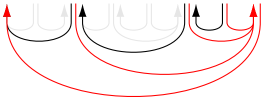

- after every iteration of line 3, we assign either nsv$[j]$ or psv$[i]$
- thus at most $2n$ element comparisons

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Computing Next Smaller Suffixes

Require: String $S[1..n]$
Ensure: Arrays nss and pss

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3: $\quad$ **while** $j > 0 \land S_j \succ S_i$ **do**
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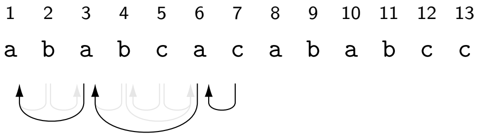

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Require: String $S[1..n]$
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1	2	3	4	5	6	7	8	9	10	11	12	13
a	b	a	b	c	a	c	a	b	a	b	c	c

- after every iteration of line 3, we assign either nss[j] or pss[i]
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```

1	2	3	4	5	6	7	8	9	10	11	12	13
a	b	a	b	c	a	c	a	b	a	b	c	c

$$\#\binom{\text{comparisons}}{\text{for } i{=}8} =$$

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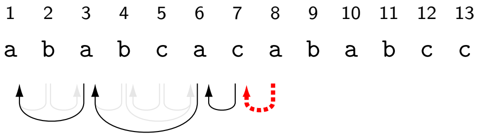

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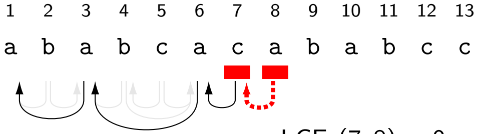

$\text{LCE}_r(7, 8) = 0$

$\# \binom{\text{comparisons}}{\text{for } i=8} =$

- after every iteration of line 3, we assign either nss[j] or pss[i]
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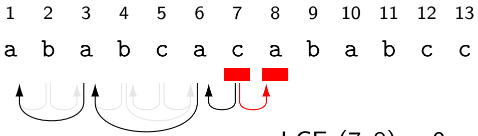

$\mathrm{LCE}_r(7, 8) = 0$

$\#\binom{\text{comparisons}}{\text{for } i{=}8} = 1 +$

- after every iteration of line 3, we assign either nss[j] or pss[i]
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$$\#\binom{\text{comparisons}}{\text{for } i{=}8} = 1 +$$

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1	2	3	4	5	6	7	8	9	10	11	12	13
a	b	a	b	c	a	c	a	b	a	b	c	c

$\mathrm{LCE}_r(6, 8) = 1$

$\#\binom{\text{comparisons}}{\text{for } i{=}8} = 1 +$

- after every iteration of line 3, we assign either nss[j] or pss[i]
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Computing Next Smaller Suffixes

Require: String $S[1..n]$
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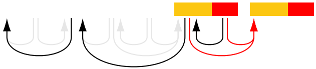
a	b	a	b	c	a	c	a	b	a	b	c	c



$\mathrm{LCE}_r(6, 8) = 1$

$$\#\binom{\text{comparisons}}{\text{for } i{=}8} = 1 + 2 +$$

- after every iteration of line 3, we assign either $\mathsf{nss}[j]$ or $\mathsf{pss}[i]$
- thus at most $2n$ suffix comparisons
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Computing Next Smaller Suffixes

Require: String $S[1..n]$
Ensure: Arrays nss and pss

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1: for i = 1 to n do
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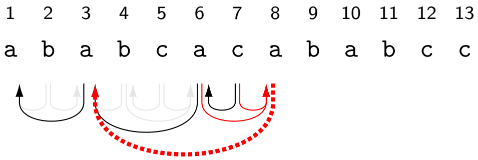

$$\#\binom{\text{comparisons}}{\text{for } i{=}8} = 1 + 2 +$$

- after every iteration of line 3, we assign either nss$[j]$ or pss$[i]$
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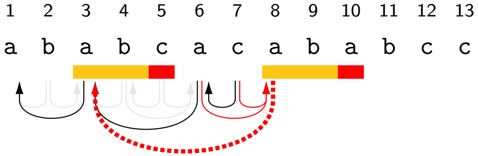

$$\mathsf{LCE}_r(3,8) = 2$$

$$\#\binom{\text{comparisons}}{\text{for } i{=}8} = 1 + 2 +$$

- after every iteration of line 3, we assign either nss[j] or pss[i]
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Computing Next Smaller Suffixes

Require: String $S[1..n]$
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1: for i = 1 to n do
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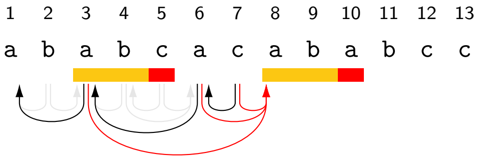

$\mathrm{LCE}_r(3, 8) = 2$

$\#\binom{\text{comparisons}}{\text{for } i=8} = 1 + 2 + 3 +$

- after every iteration of line 3, we assign either nss$[j]$ or pss$[i]$
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Computing Next Smaller Suffixes

Require: String $S[1..n]$
Ensure: Arrays nss and pss

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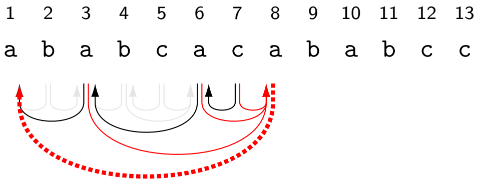

$$\#\binom{\text{comparisons}}{\text{for } i{=}8} = 1 + 2 + 3 +$$

- after every iteration of line 3, we assign either nss$[j]$ or pss$[i]$
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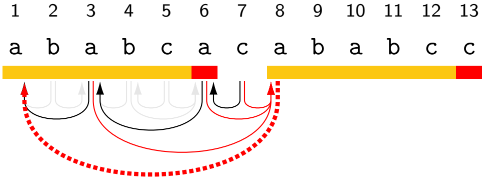

$$\mathrm{LCE}_r(1, 8) = 5$$

$$\#\binom{\text{comparisons}}{\text{for } i{=}8} = 1 + 2 + 3 +$$

- after every iteration of line 3, we assign either nss$[j]$ or pss$[i]$
- thus at most $2n$ suffix comparisons
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Computing Next Smaller Suffixes

Require: String $S[1..n]$
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1: for i = 1 to n do
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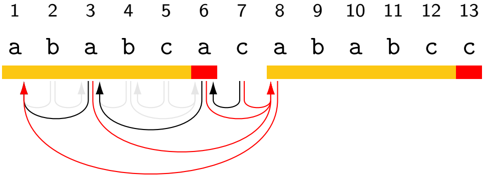

$$\mathrm{LCE}_r(1,8) = 5$$

$$\#\binom{\text{comparisons}}{\text{for } i=8} = 1 + 2 + 3 + 6 = 12$$

- after every iteration of line 3, we assign either nss[j] or pss[i]
- thus at most $2n$ suffix comparisons
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Computing Next Smaller Suffixes

Require: String $S[1..n]$
Ensure: Arrays nss and pss

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1: for i = 1 to n do
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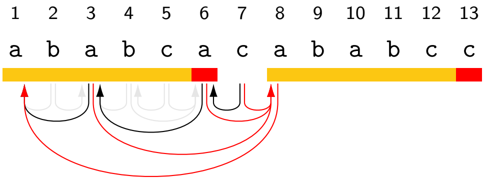
6:     pss[i] ← j
```

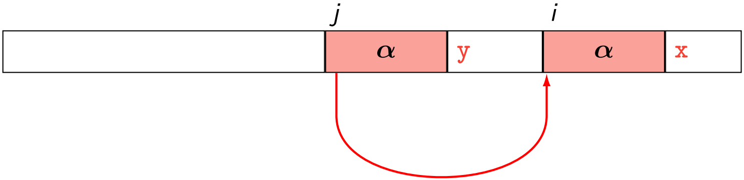


$$\mathrm{LCE}_r(1,8) = 5$$

$$\#\binom{\text{comparisons}}{\text{for } i{=}8} = 1 + 2 + 3 + 6 = 12$$

- after every iteration of line 3, we assign either nss[j] or pss[i]
- thus at most $2n$ suffix comparisons
    - but possibly many more symbol comparisons
    - some strings require $\Omega(n^2)$ symbol comparisons

Skipping Symbol Comparisons (fixed *i*)



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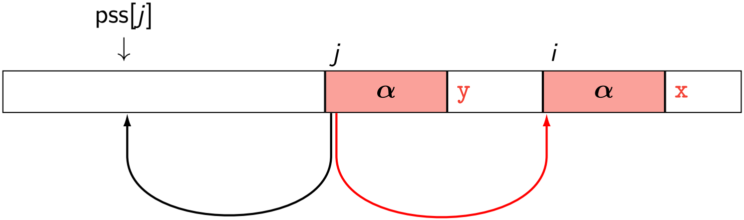

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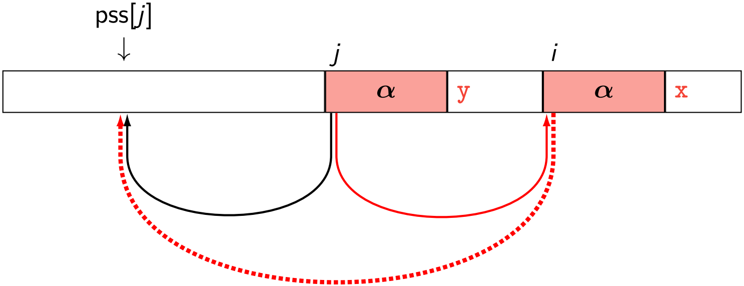

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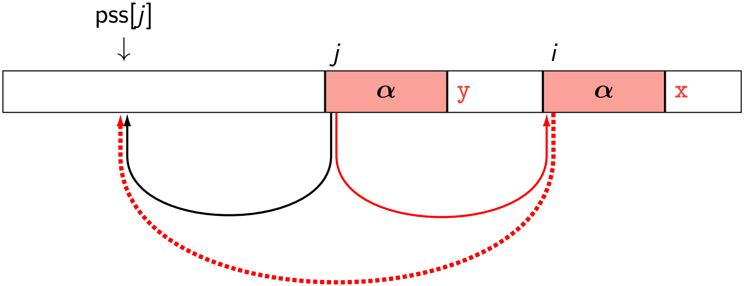

- store the computed LCEs together with edges

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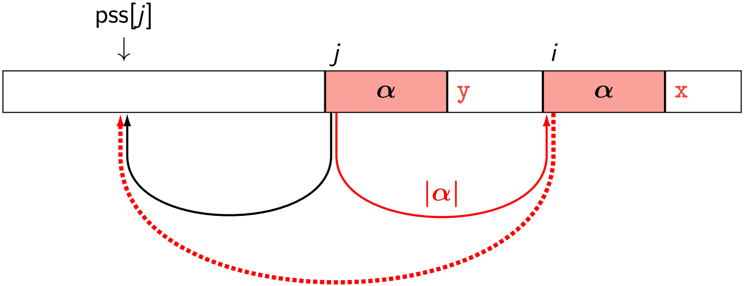

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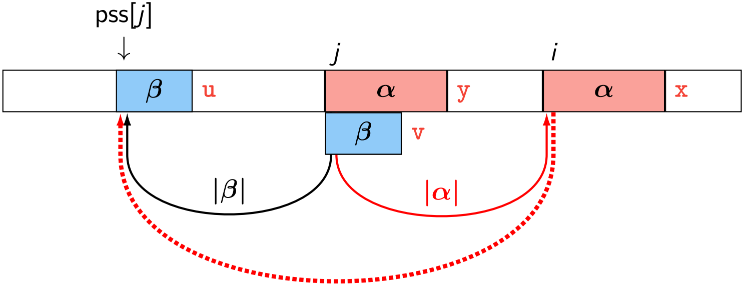

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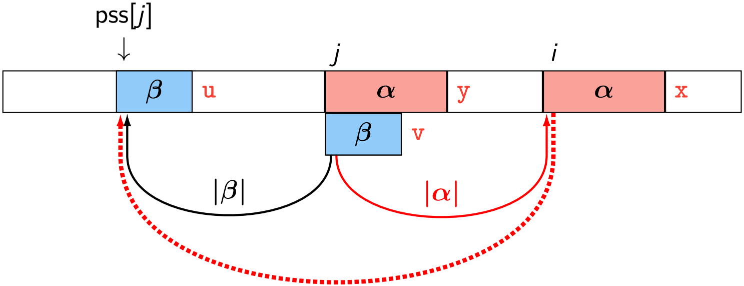
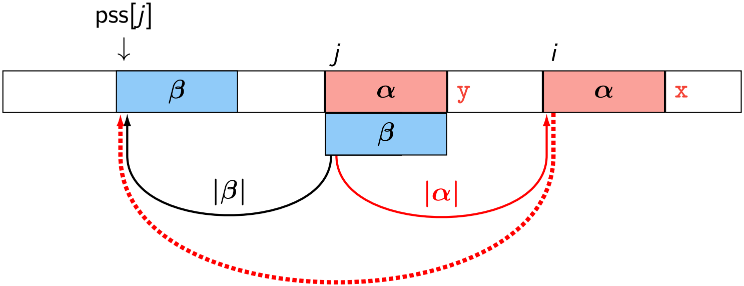


- store the computed LCEs together with edges
  - if $|\beta| < |\alpha|$, then $\mathrm{LCE}_r(pss[j], i) = |\beta|$ and $\mathrm{pss}[i] = \mathrm{pss}[j]$

Skipping Symbol Comparisons (fixed *i*)



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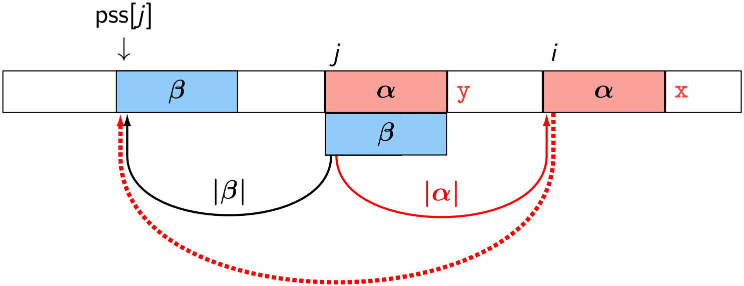

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  - if $|\beta| \geq |\alpha|$, then $\mathrm{LCE}_r(pss[j], i) \geq |\alpha|$

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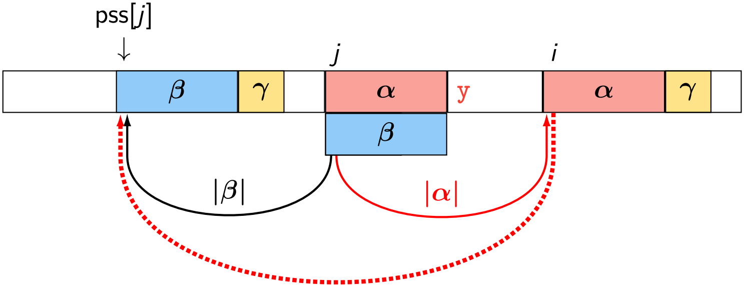

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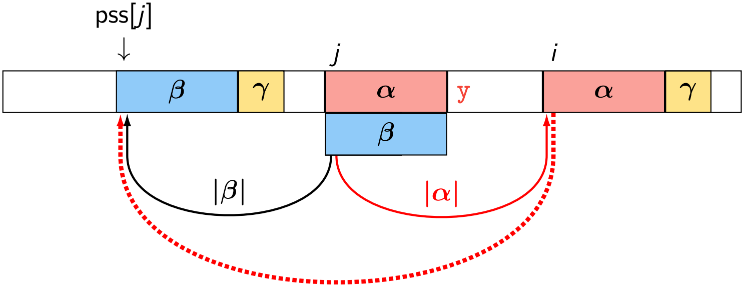

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  - if $|\beta| \geq |\alpha|$, then $\mathrm{LCE}_r(pss[j], i) \geq |\alpha|$
- still $\Omega(n^2)$ comparisons in the worst case

Skipping Symbol Comparisons (generally)

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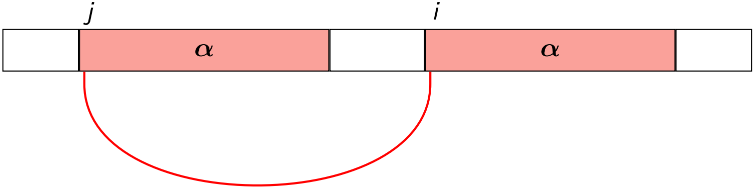

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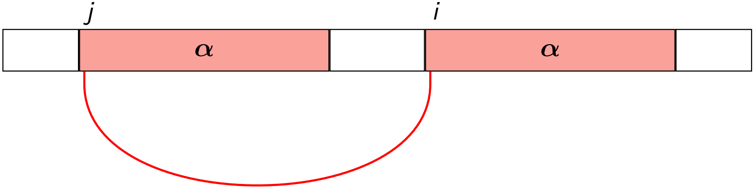
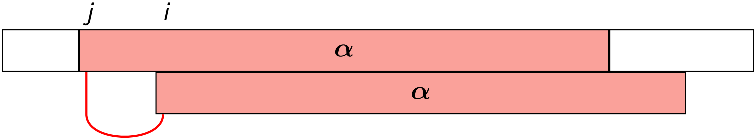


- possible, but not a problem:



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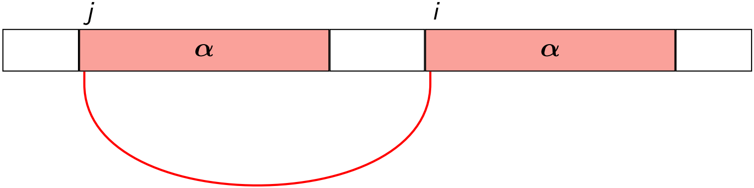

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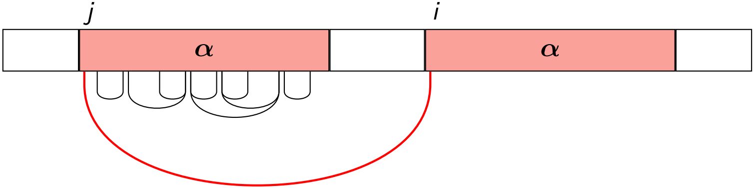
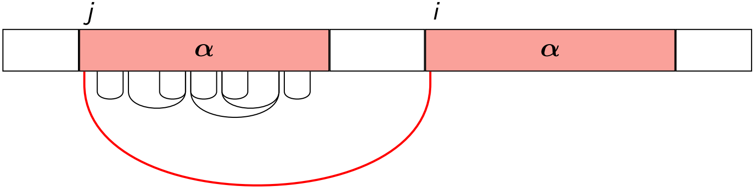


- consider edges in left occurrence of α in computational order

Skipping Symbol Comparisons (generally)



- consider edges in left occurrence of α in computational order
- find first edge with "long" LCE

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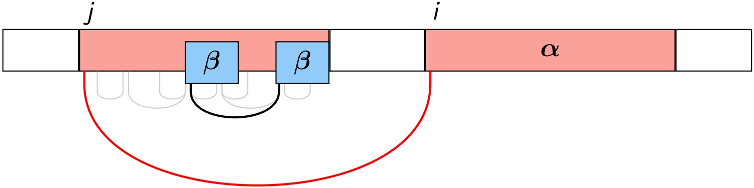

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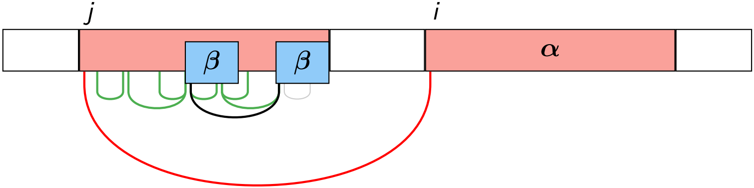

- consider edges in left occurrence of α in computational order
- find first edge with "long" LCE
- transfer all previous edges to right occurrence of α

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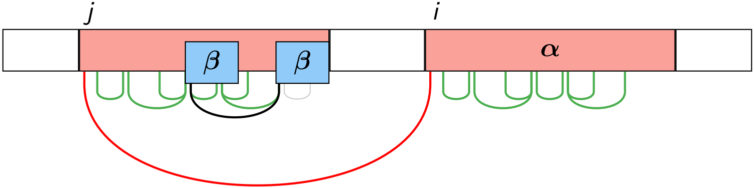

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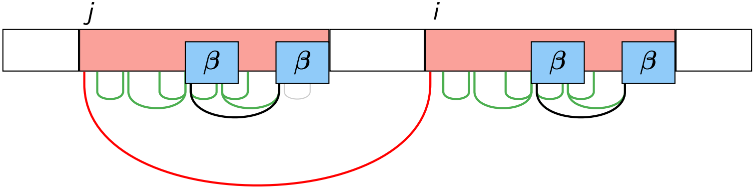

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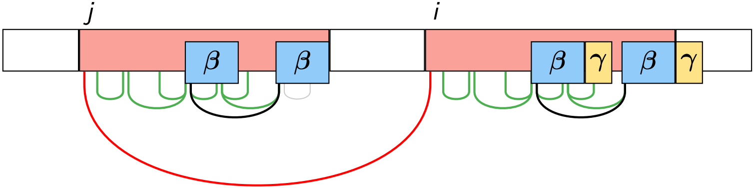

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Computing the LCEs in the Left Direction

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- consider the NSS edges in decreasing order of start positions

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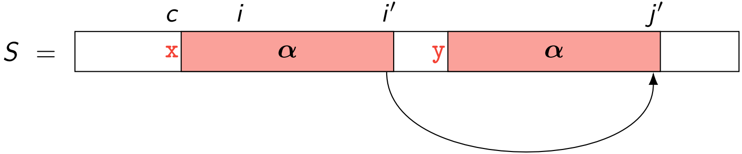

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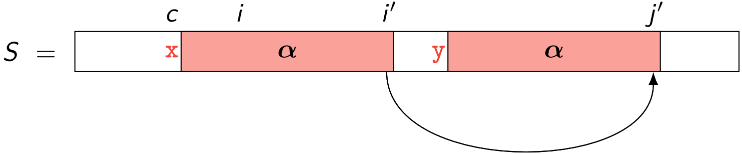
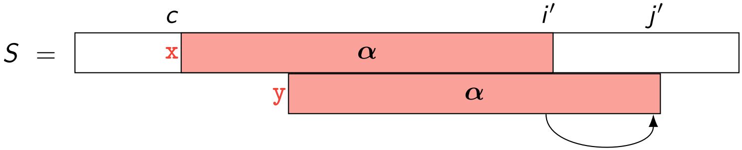


- in the paper:



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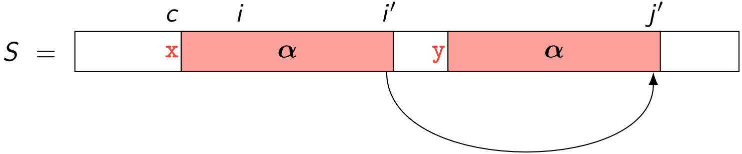

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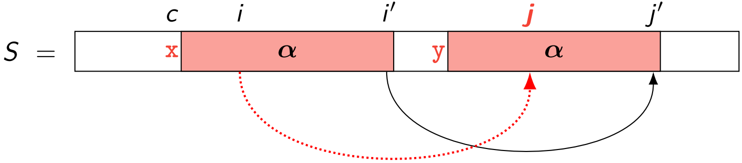

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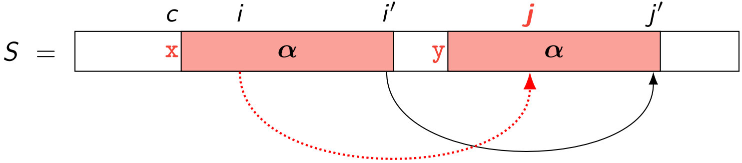

$S_j \prec S_i$

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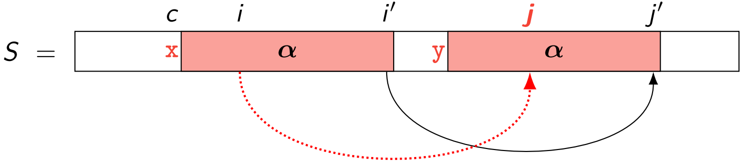

$$S_j \prec S_i \prec S_{i'}$$

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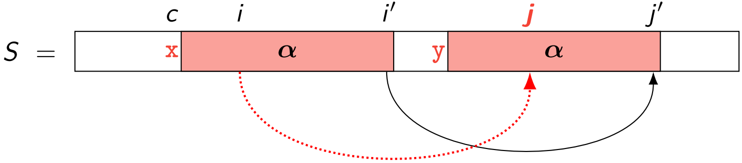

$$S_j \prec S_i \prec S_{i'} \implies \mathsf{nss}[i'] \leq j$$ ↯

Property 1:

$\mathsf{nss}[i] \notin (i', j')$

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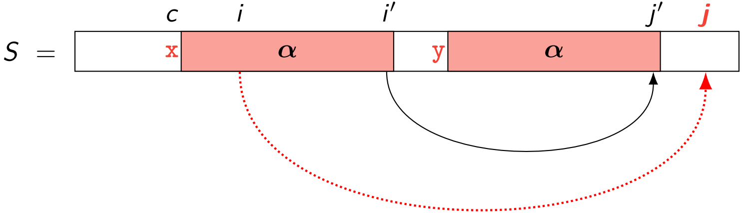

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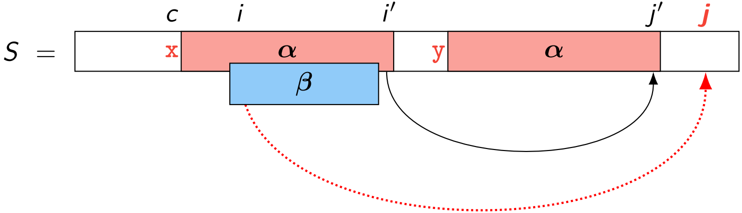

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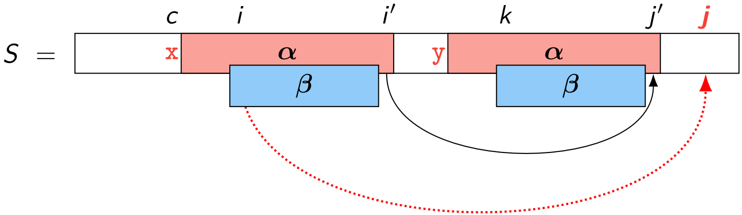

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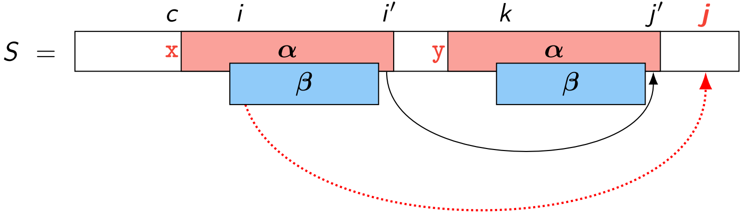

$S_{i'} \succ S_{j'}$

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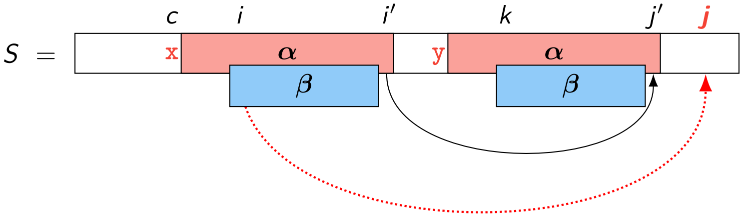

$$S_{i'} \succ S_{j'}$$
$$\iff \beta S_{i'} \succ \beta S_{j'}$$

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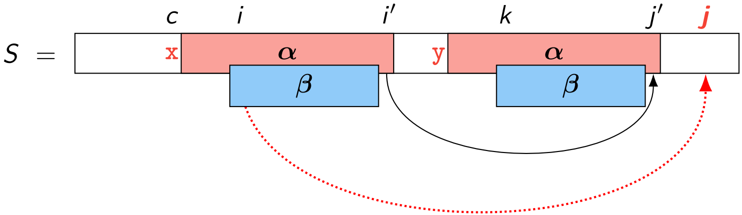

$$S_{i'} \succ S_{j'}$$
$$\iff \beta S_{i'} \succ \beta S_{j'}$$
$$\iff S_i \succ S_k$$

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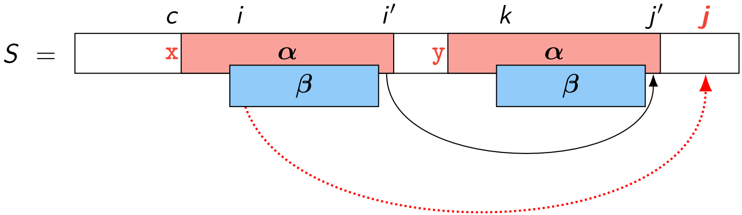

$$
\begin{aligned}
& S_{i'} \succ S_{j'} \\
\iff & \beta S_{i'} \succ \beta S_{j'} \\
\iff & S_i \succ S_k \\
\implies & \mathrm{nss}[i] \leq k \ \lightning
\end{aligned}
$$

Property 1:

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Property 2:

$\mathrm{nss}[i] \notin [j', n]$

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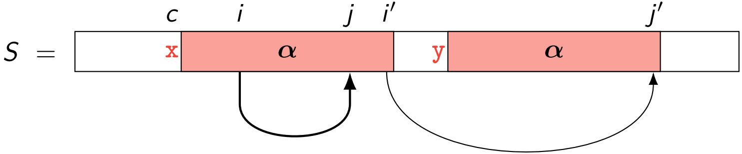

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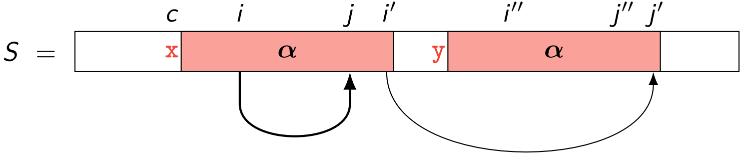

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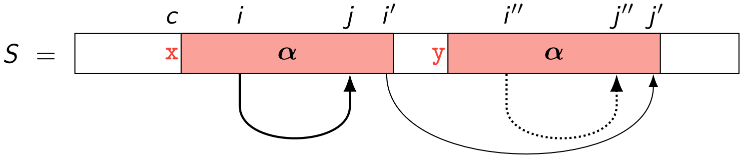

Property 1:
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Property 3:
$\mathrm{nss}[\underbrace{i - i' + j'}_{=i''}] = \underbrace{j - i' + j'}_{=j''}$

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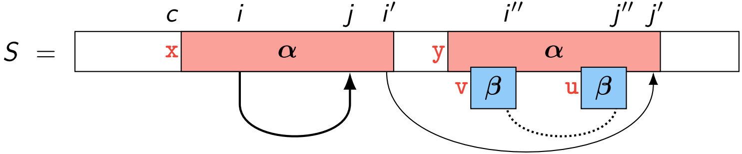

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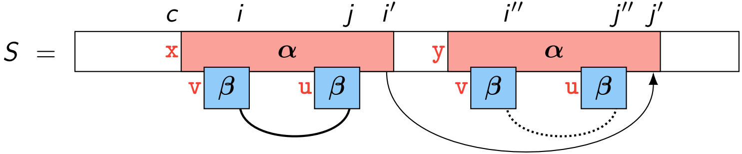

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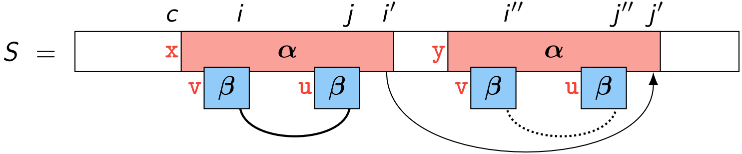

- "short" LCE between i'' and j'' $\implies$ $\mathrm{LCE}_\ell(i,j) = \mathrm{LCE}_\ell(i'',j'')$

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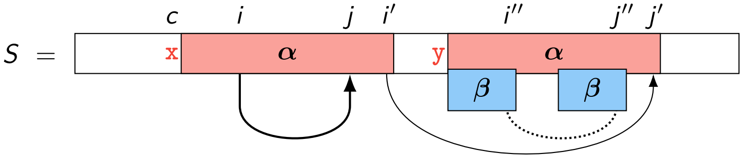

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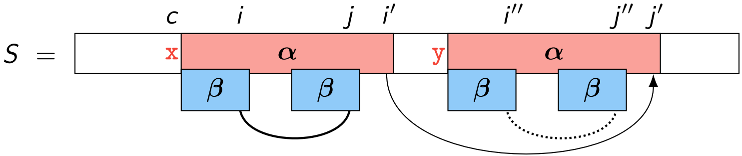

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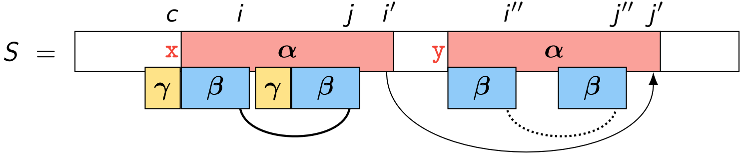

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Property 1:

$\mathrm{nss}[i] \notin (i', j')$

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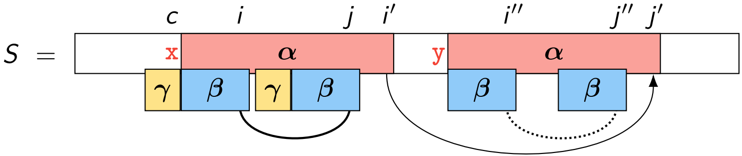
$$\mathrm{nss}[\underbrace{i - i' + j'}_{=i''}] = \underbrace{j - i' + j'}_{=j''}$$

Computing the LCEs in the Left Direction

- consider the NSS edges in decreasing order of start positions
- keep track of the leftmost inspected symbol at position c



- "short" LCE between i'' and j'' $\implies$ $\mathrm{LCE}_\ell(i,j) = \mathrm{LCE}_\ell(i'',j'')$
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Computing the LCEs in the Left Direction

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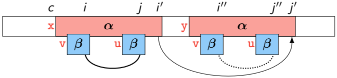

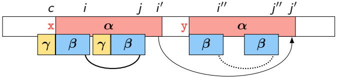

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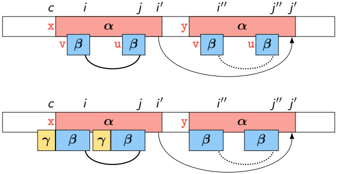

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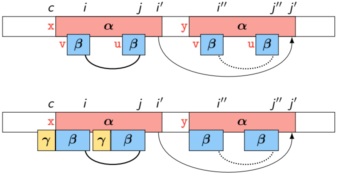

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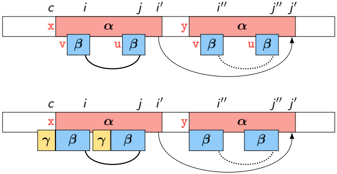

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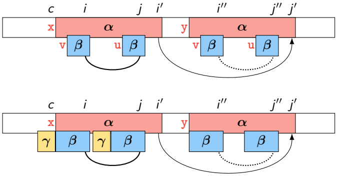

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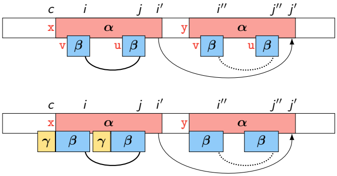

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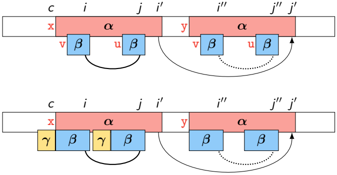

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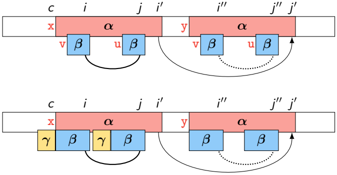

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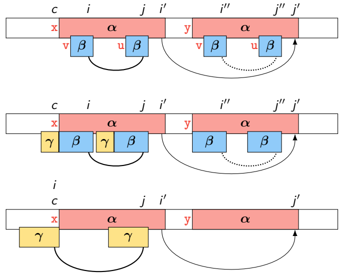

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            c ← i − LCE_ℓ(i, j)   // = c − LCE_ℓ(c, j − (i − c))
            d ← j − i
```

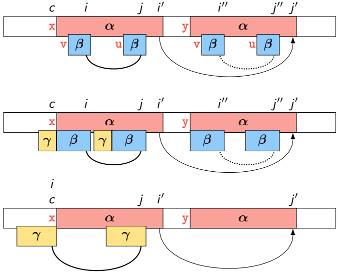

Preliminary Experiments

- AMD EPYC 7452 CPU
- repeat each experiment five times, use median as result
- Texts from `http://pizzachili.dcc.uchile.cl/`

Text n in MiB	**sources** 201 MiB	**pitches** 53 MiB	**proteins** 1024 MiB	**dna** 385 MiB	**english** 1024 MiB	**xml** 282 MiB	**fib41** 255 MiB	**tm29** 256 MiB	**rs.13** 206 MiB
runs/100n									
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runs/100n	4.7	11.7	7.0	25.3	2.4	3.4			
MiB/s	11.4	11.0	10.9	8.8	10.5	12.8			

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runs/100n	4.7	11.7	7.0	25.3	2.4	3.4	76.3	83.3	92.7
MiB/s	11.4	11.0	10.9	8.8	10.5	12.8	15.4	15.6	15.1

Conclusion

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Thanks for your attention! Questions?