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Abstract. Every drawing of a non-planar graph G in the plane induces a *planarization*, i.e., a planar graph obtained by replacing edge crossings with dummy vertices. In this paper, we consider the relationship between the capacity of a minimum *st*-cut in a graph G and its planarizations. We show that these capacities need not be equal. On the other hand, we prove that every crossing minimal planarization can be efficiently transformed into another crossing minimal planarization that preserves the capacity of a minimum *st*-cut in G . Furthermore, we extend the result to general (reasonable) planarizations.

This property turns out to be a powerful tool for reducing the computational efforts in crossing minimization algorithms. Another application is the correction of a proof given by Širáň [8], that shows an additivity property of the crossing number with respect to certain decompositions.

1 Introduction

A drawing of a graph G on the plane is an injection of the vertices of G to points in the plane and a mapping of the edges to simple continuous curves between the images of their endpoints, without containing the image of any other vertex. Any point other than the images of the vertices may only be contained in at most two curves. Such a point which is contained in exactly two curves is called a *crossing*.

The *crossing minimization problem* is a prominent optimization problem in graph theory. Given a graph G , we want to find a drawing of G in the plane with the minimum number of edge crossings. The minimum such number is called the *crossing number* of G , denoted with $\text{cr}(G)$. Garey and Johnson [6] showed that crossing minimization is NP-hard, but the approximability status is still unclear. A vast amount of publications deal with various aspects of the problem, including bounds for the crossing number and heuristic solution approaches; see e.g., [9]. Only recently, exact methods based on branch-and-cut have been proposed; see [1]. However, only small instances can be solved in reasonable time. Therefore, recent results try to reduce the size of the graph before applying exact or heuristic crossing minimization methods, e.g., using decomposition strategies such as in [7].

In this paper, we deal with graph theoretic properties of the planar graph induced by a drawing of G . This graph is called *planarization*, and is obtained by replacing the edge crossings in the drawing by so-called *crossing vertices*. Many graph properties are trivially preserved by a planarization, e.g., connectivity and

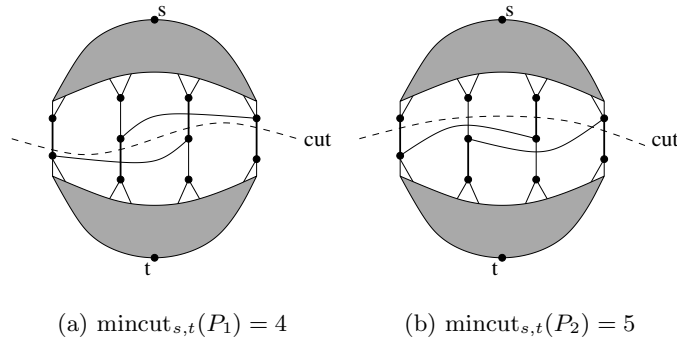


Fig. 1: Two crossing minimal planarizations of a graph G with $\text{mincut}_{s,t}(G) = 4$. The gray areas represent triconnected subgraphs, and the thick lines are the cut edges of G . The dashed lines labeled *cut* mark the minimum st -cut of each planarization.

biconnectivity. On the other hand, though a cut vertex is also a cut vertex in a crossing minimal planarization, this does not hold in general for a separation pair; see [7]. We restrict our attention to minimum st -cuts, i.e., the smallest sets of edges whose deletion disconnects the vertices s and t . The cardinality of such an edge set is called the *capacity* of the cut, denoted by $\text{mincut}_{s,t}(G)$. One might take for granted that the capacity of a minimum st -cut of G is always preserved by a crossing minimal planarization, but this is indeed not the case:

Observation 1 *A crossing minimal planarization of G can have a larger minimum st -cut than G .*

Figure 1 shows two planarizations P_1 and P_2 of the same graph G , both with exactly two crossings, which is the smallest number possible. The capacity of a minimum st -cut in G is 4. The minimum st -cut of P_1 has the same capacity, but the minimum st -cut of P_2 has capacity 5. In fact, this example reveals basic structures occurring in planarizations with larger minimum st -cuts.

On the other hand, we can show that there is always a crossing minimal planarization preserving the capacity of a minimum st -cut. The main results (see Sect. 2) of this paper are the following theorems:

Theorem 1. *Let G be a connected graph and s and t two distinct vertices in G . There exists a crossing minimal planarization P of G with*

$$\text{mincut}_{s,t}(P) = \text{mincut}_{s,t}(G).$$

We present the proof of this theorem in Sect. 2.2. The proof is constructive in the sense that it leads to a linear time transformation algorithm (see Sect. 2.3):

Theorem 2. *Any crossing minimal planarization of G can be transformed in linear time into a crossing minimal planarization P^* of G with $\text{mincut}_{s,t}(P^*) = \text{mincut}_{s,t}(G)$.*

Furthermore, we can extend these proofs to general not-necessarily crossing minimal planarizations, as long as they are *reasonable*, i.e., each pair of edges crosses each other at most once, and no two adjacent edges cross each other. Note that it is trivial to transform any non-reasonable planarization into a corresponding reasonable planarization with at most as many crossings. We sketch the proof of the following theorem in Section 2.4:

Theorem 3. *Let G be a connected graph, s and t two distinct vertices in G , and P a reasonable planarization of G . We can transform P in linear time into a planarization P' of G with $|V(P')| \leq |V(P)|$ and $\text{mincut}_{s,t}(P') = \text{mincut}_{s,t}(G)$.*

These results turn out to be powerful tools in the context of crossing minimization. In Sect. 3, we apply our results to reduce the instance size of crossing minimization problems: in Sect. 3.1, we fix a proof given by Širáň [8] which shows the additivity of the crossing number with respect to certain decompositions; in Sect. 3.2, we present a novel technique for variable reduction in exact crossing minimization of general graphs; in Sect. 3.3 we devise a heuristic scheme to reduce the crossing number problem to 3-connected components.

2 Main Results

2.1 Preliminaries

Let $G = (V, E)$ be a connected graph. If $W \subseteq V$ then $G[W]$ denotes the subgraph induced by the vertices in W . The set of crossing minimal planarizations of G is denoted with $\Pi(G)$. Let $P \in \Pi(G)$ be a planarization of G . In order to avoid confusion, we use the term *edge segment* when we refer to an edge of P . If e is an edge of G , then $\text{seg}_P(e)$ gives the set of corresponding edge segments in P ; vice versa, $\text{seg}_P^{-1}(e')$ denotes the original edge in G of an edge segment e' .

Let $s, t \in V$. The function $\chi_{s,t}(G)$ gives the set of minimum st -cuts in G . Let $C \in \chi_{s,t}(G)$ be one of these cuts. Removing the edges in the cut disconnects G into two connected subgraphs: we denote with S_C (T_C) the vertices of the subgraph containing s (t).

2.2 Proof of Theorem 1

Theorem 1 is a direct result of the following two lemmata.

Lemma 1. *For any planarization P of G , $\text{mincut}_{s,t}(P) \geq \text{mincut}_{s,t}(G)$.*

Proof. Let D be a minimum st -cut of P . Obviously, the set $D_G := \{\text{seg}_P^{-1}(d) \mid d \in D\}$ defines an st -cut in G , and thus $\text{mincut}_{s,t}(G) \leq |D_G| \leq |D|$. \square

Lemma 2. *There exists a planarization $P \in \Pi(G)$ such that $\text{mincut}_{s,t}(G) \geq \text{mincut}_{s,t}(P)$ holds.*

In order to prove Lemma 2, we need some more definitions and lemmata. We define a set of cuts $\Xi(P, C)$, dependent on a crossing minimal planarization $P \in \Pi(G)$ and a minimum st -cut $C \in \chi_{s,t}(G)$. It contains every st -cut X of P with the following properties:

- ($\Xi/1$) The vertex sets S_X and T_X induced by the cut X are supersets of S_C and T_C , respectively;
- ($\Xi/2$) for every edge $g \in C$, X contains exactly one element of $\text{seg}_P(g)$; and
- ($\Xi/3$) X is a minimal cut with the properties ($\Xi/1$) and ($\Xi/2$). Hence all elements of $\Xi(P, C)$ have the same cardinality.

Observation 2 *The cut set $\Xi(P, C)$ is non-empty for every planarization $P \in \Pi(G)$ and every cut $C \in \chi_{s,t}(G)$.*

Observation 3 *For every cut $X \in \Xi(P, C)$, the subgraphs $P[S_X]$ and $P[T_X]$ are connected sets. (This follows from ($\Xi/3$) and the precondition that G , and therefore P as well, is connected.)*

We define a set of drawings $\Phi(P, X)$ of a planarization $P \in \Pi(G)$, depending on an st -cut $X \in \Xi(P, C)$ for some $C \in \chi_{s,t}(G)$; it contains every drawing \mathcal{D} of P with the following properties:

- ($\Phi/1$) \mathcal{D} is planar;
- ($\Phi/2$) the two vertex sets S_X and T_X reside in two disjoint regions R_S and R_T of \mathcal{D} , respectively, which are topologically equivalent to a disk; and
- ($\Phi/3$) the edge segments that connect vertices from S_X (T_X) are completely contained in the region R_S (R_T).

Lemma 3. *The set $\Phi(P, X)$ cannot be empty.*

Proof. Conceptually, we can treat a planarization P as a clustered graph C_P with two clusters exactly containing the elements of S_X and T_X , respectively. Since P is planar and connected, and S_X and T_X are connected as well, this clustered graph is *completely connected*, i.e. for each cluster ν , both ν and $\bar{\nu}$ — the graph without ν and its elements — are connected. Hence, there is a cluster planar drawing of C_P ; see [2,4]. In particular, this drawing induces a drawing of P satisfying the properties ($\Phi/1$)–($\Phi/3$). \square

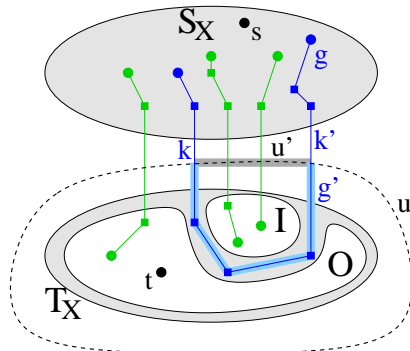


Fig. 2: The edge g generates an arch g' within T_X . We choose I and O depending on t . (Circles denote vertices, squares denote crossing vertices)

Proof of Lemma 2. An edge segment k of a cut $X \in \Xi(P, C)$ can be of one of the following two types. Let g be the original edge in G corresponding to k .

- \in : $g \in C$. By property $(\Xi/2)$, we have $\text{seg}_P(g) \cap X = \{k\}$.
- \notin : $g \notin C$. The end vertices of g belong to the same partition set induced by C , i.e., either both end vertices are in S_C or both are in T_C . Therefore all such segments k must occur pairwise, and the cardinality of $\text{seg}_P(g) \cap X$ is even.

Let k be an edge segment of type \notin . There is at least one other segment $k' \in X$ corresponding to the common original edge g . Let w.l.o.g. both end vertices of g belong to S_C . We consider a drawing from $\Phi(P, X)$. We can place a hypothetical circle u between the regions R_S and R_T in such a way, that u crosses each cut segment of X exactly once, but it does not cross any other segment (cf. Fig. 2). The aforementioned edge segments both cross u ; let k' be chosen such that the crossings of k and k' with u are successive crossings between u and g , such that the part of g between those crossing points lies in R_T . We call this part of g an *arch*, and denote it by g' .

The following lemma shows that there always exists a planarization which allows a cut X that satisfies the Ξ -properties but contains no arches. Hence it does not contain any edge segment of type \notin , which induces $|X| = \text{mincut}_{s,t}(G)$. Since the minimum st -cut of the planarization cannot be larger than X , Lemma 2 follows immediately.

Lemma 4. For any non-planar graph G and any minimum st -cut $C \in \chi_{s,t}(G)$, there is a crossing minimal planarization $P_C^* \in \Pi(G)$, such that any cut $X \in \Xi(P_C^*, C)$ contains no arches.

Proof. Let $P_C^* \in \Pi(G)$ be a planarization such that the capacities of the cuts of $\Xi(P_C^*, C)$ are minimal among all planarizations (cf. Property $(\Xi/3)$). Let us assume the existence of an arch, and let k, k', g , and g' be defined as above. The

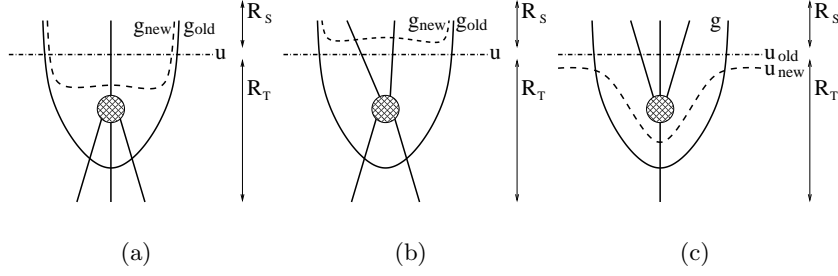


Fig. 3: Lemma 4. (a) (C1) $n_{u'} < n_{g'}$: Changing the planarization reduces the crossing number. (b) (C2) $n_{u'} = n_{g'}$: Changing the planarization reduces the cut size. (c) (C3) $n_{u'} > n_{g'}$: Changing the cut reduces the cut size, but may violate $(\Xi/2)$.

arch g' defines a bipartition of the vertices of T_C . Let $O \subset T_C$ be the subset that contains t , and $I \subset T_C$ the other one. Let \bar{O} and \bar{I} be the regions bordered by u and g' , containing O and I , respectively, and partitioning R_T . We assume w.l.o.g. that g' is an inner-most arch, i.e. \bar{I} does not contain any other arch. Let u' be the part of u that is on the boundary of \bar{I} . We count the number of crossing points between edges incident to a vertex of I and the border of \bar{I} : we denote the number of crossing points on u' by $n_{u'}$, and the number of crossing points on g' by $n_{g'}$. We distinguish by the relationship between these two values:

- C1:** $n_{u'} < n_{g'}$: We could select another routing of g' such that it crosses the same edges as u' (cf. Fig. 3(a)). This planarization would have a lower number of crossings, which is a contradiction to the crossing minimality of P_C^* .
- C2:** $n_{u'} = n_{g'}$: We could select another routing of g' , just as in case C1. The planarization would have the same number of crossings, but the cut would be smaller, since the arch g' would be in R_S and therefore not part of the cut (cf. Fig. 3(b)). This would be a contradiction to the selection of P_C^* .
- C3:** $n_{u'} > n_{g'}$: By changing the selection of cut segments (represented by rerouting u , cf. Fig. 3(c)), we could generate a smaller cut X' , but in general there does not exist any cut $C' \in \chi_{s,t}(G)$ for which X' would satisfy the property $(\Xi/2)$. Furthermore, the selection of cut edges alone is not sufficient to retrieve a cut as small as the cut through G itself (e.g., in Fig. 1; note that the corresponding Ξ -cut for the second planarization has cardinality 8).

We know that more edges incident to I leave the region \bar{I} over u' than over g' . But since the vertices of I are elements of T_C , the majority of the end vertices of these outbound edges belong to T_C . (Note that there cannot exist a balanced situation, since this would imply that all vertices of I could belong to some $S_{\bar{X}}$).

Therefore there are at least $\lfloor (n_{u'} - n_{g'})/2 \rfloor + 1$ arches which reside in R_S , before the according edges enter R_T over u (cf. Fig. 4). All these arches fall

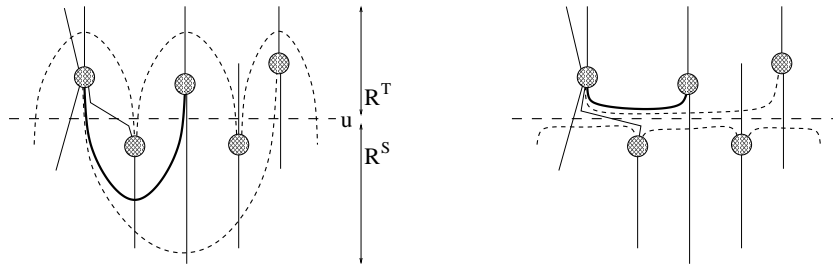


Fig. 4: ($C3$) $n_{u'} > n_{g'}$: A chain of arches. Changing the planarization (right) reduces either the crossing number or the cut size.

into case $C3$ as well, since we already showed that the cases $C1$ and $C2$ lead to contradictions. Hence there are sets of causally connected arches:

Let $K(g')$ be a set of arches g'_i ($1 \leq i \leq |K(g')|$) with the original edges $g_i \in E(G)$. We denote the thereby induced parts of u by u'_i . We call $K(g')$ a *chain*, if it is the minimal set that satisfies the following properties:

- g' is an element of $K(g')$.
- An arch g'_j is an element of $K(g')$, if it touches at least one u'_i ($1 \leq i \leq |K(g')|$).

Let $m_{u'_i}$ denote the number of crossing points on u'_i induced by type \in segments corresponding to edges incident to vertices of I . We know for each arch in $K(g')$ that $n_{g'_i} \geq m_{u'_i}$, due to the crossing minimality of P_C^* .

We apply the transformation as described in case $C1$ simultaneously on all arches of the chain. On each arch g'_i only $m_{u'_i}$ crossings will remain. We can distinguish between two sub-cases:

$C3a$: $n_{g'_i} > m_{u'_i}$ for at least one i : the crossing number would decrease, which is a contradiction to the crossing minimality of P_C^* , as in case $C1$.

$C3b$: $n_{g'_i} = m_{u'_i}$ for every arch: the transformation would unveil a contradiction to the minimality of X , as in case $C2$. \square

We showed that we can always select an appropriate $P_C^* \in \Pi(G)$ for any $C \in \chi_{s,t}(G)$, such that each thereby induced cut $X \in \Xi(P_C^*, C)$ does not contain any type \notin segments. This means that X contains exactly $\text{mincut}_{s,t}(G)$ elements of type \in , and therefore $\text{mincut}_{s,t}(P_C^*) \leq \text{mincut}_{s,t}(G)$.

2.3 A Linear Time Transformation Algorithm

The case differentiation in the proof of Lemma 4 implies a transformation algorithm for any crossing minimal planarization P of G in a straight-forward way. Due to the crossing minimality of P , there will only be arches falling into the categories $C2$ and $C3b$. Hence it is possible to run along the hypothetical circle u and check the cutting segments whether they are of type \in or \notin . Since this check takes constant time, the complete testing loop takes $O(m)$ time, where m

is the number of edges in P . Whenever a type \neq segment occurs, the according arch is transformed as stated in the proof above, i.e., it is rerouted through its own region. Since each segment is modified at most once, all these transformations together will take at most $O(m)$ time. No explicit analysis of chains is required. Hence we get Theorem 2 (see Sect. 1).

2.4 Extension to Reasonable Planarizations

The proofs in the above sections can be generalized for not-necessarily crossing minimal, reasonable planarizations, obtaining Theorem 3. Due to space constraints, we will only outline the corresponding proof within this section. Its main ideas and techniques are similar to the proofs above.

Let P be a reasonable planarization of G and s, t two distinct vertices in G . Obviously, $\text{mincut}_{s,t}(P) \geq \text{mincut}_{s,t}(G)$, hence we need to proof that we can create P' in linear time, such that $\text{mincut}_{s,t}(P') \leq \text{mincut}_{s,t}(G)$ without increasing the number of crossings. We extend our definitions of the set Ξ to a set Ξ' for reasonable planarizations, and will use that set in the remainder of the proof, substituting Ξ . Each $X \in \Xi'(P, C)$ is an st -cut in P satisfying the properties defined for the set Ξ , where $(\Xi/2)$ is substituted by the following weaker version:

$(\Xi'/2)$ For every edge $g \in C$, X contains at least one element of $\text{seg}_P(g)$; X is a minimal cut satisfying this property, i.e., it contains as few segments violating the strict $(\Xi/2)$ -property as possible.

Similar to the proof of Lemma 2, we investigate drawings of $\Phi(P, X)$ with $X \in \Xi'(P, C)$. The weakening of property $(\Xi'/2)$ leads to edges connecting a vertex of T_C with a vertex of S_C , but crossing the hypothetical circle u multiple times. Let g be such an edge. Obviously g crosses u an odd number of times, and thereby introduces a new type of arches.

The analysis of arches in Lemma 4 gave three different cases, the third of which unfolded into two subcases. Although we now have a new type of arches to consider, careful case distinction shows that we basically obtain the same differentiation. Out of these four cases, two cases ($C2$ and $C3b$) induce a scheme of shifting the arch a – i.e., generating a related planarization P' – such that a is not part of any new Ξ' -cut X' anymore; the number of crossings in P' does not increase thereby. This transformation is identical to the one necessary for Theorem 2. The two other cases ($C1$ and $C3a$) were shown not to exist for crossing minimal planarizations. Although they can occur for non-crossing-minimal planarizations, the described shifting scheme always guarantees the following important properties for the newly generated planarization P' , which can be easily verified: (a) the number of crossing vertices in P' is less than in P , (b) the considered arches are not part of any new Ξ' -cut X' , and (c) no new segments are added to X' compared to X . The linear time complexity of these shifting schemata can be argued as in Section 2.3. Hence we obtain Theorem 3.

3 Applications

3.1 Crossing Number and st -Decomposition

Širáň [8] studied the crossing number of a graph with respect to decomposing the graph into two subgraphs having exactly two vertices in common. Let G be a graph with two distinct vertices s and t . We call (H, K) an st -decomposition of G if $H \cup K = G$ and $H \cap K = [s, t]$. Širáň showed that

$$\text{cr}(G) \geq \text{cr}(H) + \text{cr}(K^*), \quad (1)$$

where K^* is obtained from K by adding $\text{mincut}_{s,t}(H)$ many (s, t) -edges; see Lemma 1 in [8]. Širáň wanted to prove that equality holds if $\text{cr}(H) = \text{cr}(H \cup (s, t))$. However, using the results of this paper, we can show that his proof is not correct. On the other hand, we can fix the proof showing that his result is still true. We first state the theorem.

Theorem 4 (Theorem 1 in [8]). *Let (H, K) be an st -decomposition of G such that $\text{cr}(H) = \text{cr}(H \cup (s, t))$, and let K^* be the graph obtained from K by adding $\lambda := \text{mincut}_{s,t}(H)$ many (s, t) -edges. Then, $\text{cr}(G) = \text{cr}(H) + \text{cr}(K^*)$.*

Širáň's construction requires to find a crossing minimal drawing \mathcal{D}_H of H with both s and t on the external face in which it is possible to separate s and t by a line that only crosses λ edges. Let Π be the embedding of the planarization P_H induced by \mathcal{D}_H in which the edge $e_{st} = (s, t)$ is inserted into the external face. Then, we have to find a path of length λ in the dual graph of Π connecting the two faces adjacent to e_{st} without using the dual edge of e_{st} . It has been shown in [7] that the minimum length of such a path is the capacity of a minimum st -cut. However, by Observation 1, $\text{mincut}_{s,t}(P_H)$ may be larger than $\text{mincut}_{s,t}(H)$; cf. Fig. 1. Širáň assumed that any crossing minimal drawing of H has a traversing path with respect to (s, t) of length λ . Since this is not the case, his proof is not correct. On the other hand, Theorem 1 proves the existence of such a crossing minimal drawing with the required properties, thus fixing the proof.

3.2 Reduction of Variables in Exact Crossing Minimization

Since crossing minimization is an NP-hard problem, using exact algorithms is expected to be feasible only for relatively small graphs. Therefore a reduction of the involved variables can improve the practical applicability of such algorithms dramatically. Let C be an arbitrary minimum st -cut of G for some vertices s and t . By Lemma 4, there exists a crossing minimal planarization P_C^* which contains no arches, and thus no type \notin segments. This implies the following corollary:

Corollary 1. *Let G be a graph with two distinct vertices s and t , and let C be any minimum st -cut of G . Then, there exists a crossing minimal drawing of G that contains no crossing between an edge of $G[S_C]$ with an edge of $G[T_C]$.*

Hence we can search for the *best-balanced minimum st -cut*, i.e., a pair $(s, t) \in V(G) \times V(G)$ with a minimum st -cut C^* which minimizes the *balance function* $b(C^*) := ||E(G[S_{C^*}])| - |E(G[T_{C^*}])||$, i.e., the difference in the cardinality of the edge sets of the induced partitions. C^* does not have to be the minimum st -cut over all (s, t) pairs. This problem differs from the NP-hard problem known as *minimum graph bisection* (see, e.g., [5]), which minimizes the cut over all perfect partitionings.

Having the cut C^* , we can forbid that any edge of $G[S_{C^*}]$ crosses an edge of $G[T_{C^*}]$, and still obtain an optimal solution. Note that we can apply the corollary for any minimum st -cut; using the best-balanced one only maximizes the gain. Consider the integer linear program to solve the crossing minimization problem optimally, presented in [1]. It needs a 0/1 variable for each pair of edges, representing their crossing. By applying our corollary, we can – asymptotically for large graphs and relatively small cuts – reduce the number of these variables by up to 50%. But note that we cannot guarantee such an improvement.

3.3 Merging of 3-connected Components

The computational effort for (optimal) crossing minimization can be reduced by applying a special preprocessing technique called *non-planar core reduction* [7]. This method analyzes the SPQR-tree [3] of the given non-planar graph and cuts off certain planar components attached to the rest of the graph only at some vertices s and t . These can later be reintroduced based on their minimum st -cuts.

Due to Theorem 3 it is reasonable to calculate the minimum st -cut of a non-planar subgraph without knowing the exact planarization, nor its crossing number. Hence, our transformation algorithm enables us to devise a heuristic to solve the minimization problems on each non-planar triconnected component separately, and merge them afterwards. However, that strategy might not achieve optimality, but we expect it to be close to the optimal solution. Furthermore, we could be able to further simplify the core by carefully extracting special non-planar subgraphs, and being able to reintroduce them optimally later on.

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