Computational Intelligence

Winter Term 2009/10

Prof. Dr. Günter Rudolph
Lehrstuhl für Algorithm Engineering (LS 11)
Fakultät für Informatik
TU Dortmund

- Fuzzy relations
- Fuzzy logic
  - Linguistic variables and terms
  - Inference from fuzzy statements

relations with conventional sets $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$:

$$R(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n) \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$$

notice that cartesian product is a **set**!
⇒ all set operations remain valid!

crisp membership function (of x to relation R)

$$R(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } (x_1, x_2, \ldots, x_n) \in R \\ 0 & \text{otherwise} \end{cases}$$

Definition

Fuzzy relation = fuzzy set over crisp cartesian product $\mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$ ■

→ each tuple $(x_1, \ldots, x_n)$ has a degree of membership to relation

→ degree of membership expresses
strength of relationship between elements of tuple

appropriate representation: n-dimensional membership matrix

example: Let X = { New York, Paris } and Y = { Bejing, New York, Dortmund }.

relation R = "very far away"

membership matrix →

relation R	New York	Paris
Bejing	1.0	0.9
New York	0.0	0.7
Dortmund	0.6	0.3

Definition

Let R(X, Y) be a fuzzy relation with membership matrix $\mathbb{R}$. The ***inverse fuzzy relation*** to R(X,Y), denoted $R^{-1}(Y, X)$, is a relation on Y x X with membership matrix $\mathbb{R}^{-1} = \mathbb{R}'$. ■

Remark: $\mathbb{R}'$ is the transpose of membership matrix $\mathbb{R}$.

Evidently: $(\mathbb{R}^{-1})^{-1} = \mathbb{R}$ since $(\mathbb{R}')' = \mathbb{R}$

Definition

Let P(X, Y) and Q(Y, Z) be fuzzy relations. The operation ∘ on two relations, denoted P(X, Y) ∘ Q(Y, Z), is termed ***max-min-composition*** iff

$$R(x, z) = (P \circ Q)(x, z) = \max_{y \in Y} \min \{ P(x, y), Q(y, z) \}.$$ ■

Theorem

a) max-min composition is associative.

b) max-min composition is not commutative.

c) $(P(X,Y) \circ Q(Y,Z))^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y, X)$.

membership matrix of max-min composition
determinable via "fuzzy matrix multiplication": R = P ∘ Q

fuzzy matrix multiplication $r_{ij} = \max_k \min\{p_{ik}, q_{kj}\}$

crisp matrix multiplication $r_{ij} = \sum_k p_{ik} \cdot q_{kj}$

further methods for realizing compositions of relations:

max-prod composition

$$(P \odot Q)(x, z) = \max_{y \in \mathcal{Y}}\{P(x, y) \cdot Q(y, z)\}$$

<u>generalization:</u> **sup-t composition**

$$(P \circ Q)(x, z) = \sup_{y \in \mathcal{Y}}\{t(P(x, y), Q(y, z))\},$$ where t(.,.) is a t-norm

e.g.:	t(a,b) = min{a, b}	⇒ max-min-composition
	t(a,b) = a · b	⇒ max-prod-composition

Binary fuzzy relations on $\mathcal{X} \times \mathcal{X}$: properties

• **reflexive**	⇔ $\forall x \in \mathcal{X}$: R(x,x) = 1
• **irreflexive**	⇔ $\exists x \in \mathcal{X}$: R(x,x) < 1
• **antireflexive**	⇔ $\forall x \in \mathcal{X}$: R(x,x) < 1
• **symmetric**	⇔ $\forall (x,y) \in \mathcal{X} \times \mathcal{X}$: R(x,y) = R(y,x)
• **asymmetric**	⇔ $\exists (x,y) \in \mathcal{X} \times \mathcal{X}$: R(x,y) ≠ R(y,x)
• **antisymmetric**	⇔ $\forall (x,y) \in \mathcal{X} \times \mathcal{X}$: R(x,y) ≠ R(y,x)
• **transitive**	⇔ $\forall (x,z) \in \mathcal{X} \times \mathcal{X}$: $R(x,z) \geq \max_{y \in \mathcal{Y}} \min \{ R(x,y), R(y,z) \}$
• **intransitive**	⇔ $\exists (x,z) \in \mathcal{X} \times \mathcal{X}$: $R(x,z) < \max_{y \in \mathcal{Y}} \min \{ R(x,y), R(y,z) \}$
• **antitransitive**	⇔ $\forall (x,z) \in \mathcal{X} \times \mathcal{X}$: $R(x,z) < \max_{y \in \mathcal{Y}} \min \{ R(x,y), R(y,z) \}$

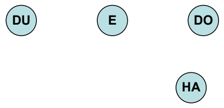
actually, here: max-min-transitivity (→ in general: sup-t-transitivity)

binary fuzzy relation on $\mathcal{X} \times \mathcal{X}$: <u>example</u>

Let $\mathcal{X}$ be the set of all cities in Germany.

Fuzzy relation R is intended to represent the concept of „very close to".

- R(x,x) = 1, since every city is certainly very close to itself.
  ⇒ **reflexive**
- R(x,y) = R(y,x): if city x is very close to city y, then also vice versa.
  ⇒ **symmetric**
- R(Dortmund, Essen) = 0.8
  R(Essen, Duisburg) = 0.7
  R(Dortmund, Duisburg) = 0.5
  R(Dortmund, Hagen) = 0.9
  ⇒ **intransitive**

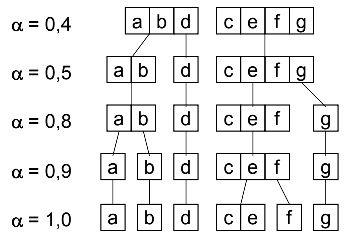


crisp:
relation R is <u>equivalence relation</u> ⇔ R reflexive, symmetric, transitive

fuzzy:
relation R is <u>similarity relation</u> ⇔ R reflexive, symmetric, (max-min-) transitive

Bsp:

	a	b	c	d	e	f	g
a	1,0	0,8	0,0	0,4	0,0	0,0	0,0
b	0,8	1,0	0,0	0,4	0,0	0,0	0,0
c	0,0	0,0	1,0	0,0	1,0	0,9	0,5
d	0,4	0,4	0,0	1,0	0,0	0,0	0,0
e	0,0	0,0	1,0	0,0	1,0	0,9	0,5
f	0,0	0,0	0,9	0,0	0,9	1,0	0,5
g	0,0	0,0	0,5	0,0	0,5	0,5	1,0



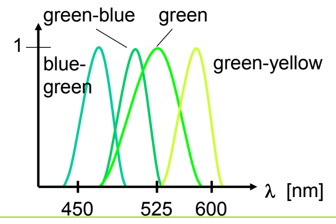
linguistic variable:

variable that can attain several values of lingustic / verbal nature

e.g.: **color** can attain values **red**, **green**, **blue**, **yellow**, …

values (red, green, …) of linguistic variable are called **linguistic terms**

linguistic terms are associated with fuzzy sets



<u>fuzzy proposition</u>

p: *temperature* is *high*

linguistic variable (LV) — linguistic term (LT)

- LV may be associated with several LT : *high, medium, low, …*
- *high, medium, low* temperature are fuzzy sets
  over numerical scale of crisp temperatures
- <u>trueness</u> of fuzzy proposition „temperature is high"
  for a given **concrete crisp** temperature value v
  is interpreted as equal to the degree of membership *high*(v)
  of the fuzzy set *high*

fuzzy proposition

p: *V* is F

linguistic variable (LV) — linguistic term (LT)

actually:

p: *V* is F(v)

and

T(p) = F(v) for a concrete crisp value v

trueness(p)

establishes connection between *degree of membership* of a fuzzy set and the *degree of trueness* of a fuzzy proposition

fuzzy proposition

p: IF *heating* is *hot*, THEN *energy consumption* is *high*

LV LT LV LT

expresses relation between

a) temperature of heating and

b) quantity of energy consumption

p: (*heating, energy consumption*) ∈ R — relation

fuzzy proposition

p: IF *X* is A, THEN *Y* is B

LV LT LV LT

How can we determine / express degree of trueness T(p) ?

- For crisp, given values x, y we know A(x) and B(y)
- A(x) and B(y) must be processed to single value via relation R
- R(x, y) = function(A(x), B(y)) is fuzzy set over X × Y
- as before: interprete T(p) as degree of membership R(x,y)

fuzzy proposition

p: IF *X* is A, THEN *Y* is B

A is fuzzy set over X

B is fuzzy set over Y

R is fuzzy set over X × Y

$\forall (x,y) \in X \times Y$: R(x, y) = Imp(A(x), B(y))

What is Imp(**·**, **·**) ?

⇒ „appropriate" fuzzy implication $[0,1] \times [0,1] \rightarrow [0,1]$

assumption: we know an „appropriate" Imp(a,b).

How can we determine the degree of trueness T(p) ?

example:

let Imp(a, b) = min{ 1, 1 – a + b } and consider fuzzy sets

A:

x_1	x_2	x_3
0.1	0.8	1.0

B:

y_1	y_2
0.5	1.0

⇒

R	x_1	x_2	x_3
y_1	1.0	0.7	0.5
y_2	1.0	1.0	1.0

z.B.
$R(x_2, y_1) = Imp(A(x_2), B(y_1)) = Imp(0.8, 0.5) = \min\{1.0, 0.7\} = 0.7$

and T(p) for (x_2, y_1) is $R(x_2, y_1) = 0.7$ ■

inference from fuzzy statements

- let $\forall x, y: y = f(x)$.

  IF X = x THEN Y = f(x)

- IF $X \in A$ THEN $Y \in B = \{ y \in Y: y = f(x), x \in A \}$

inference from fuzzy statements

- Let relationship between x and y be a relation R on X × Y

  IF X = x THEN $Y \in B = \{ y \in Y: (x, y) \in R \}$

- IF $X \in A$ THEN $Y \in B = \{ y \in Y: (x, y) \in R, x \in A \}$

inference from fuzzy statements

IF $X \in A$ THEN $Y \in B = \{ y \in Y: (x, y) \in R, x \in A \}$

also expressible via characteristic functions of sets A, B, R:

$\forall y \in Y: B(y) = \sup_{x \in X} \min \{ A(x), R(x, y) \}$

Now: A', B' fuzzy set over X resp. Y

Assume R and A' are given:

$\forall y \in Y: B'(y) = \sup_{x \in X} \min \{ A'(x), R(x, y) \}$

composition rule of inference (in matrix form): **$B' = A' \circ R$**

inference from fuzzy statements

- conventional: modus ponens

$a \Rightarrow b$
a
———
b

- fuzzy: generalized modus ponens (GMP)

IF *X* is A, THEN *Y* is B
X is A'
———
Y is B'

e.g.: IF *heating* is hot, THEN *energy consumption* is high
heating is warm
———
energy consumption is normal

example: GMP

consider A:

x_1	x_2	x_3
0.5	1.0	0.6

B:

y_1	y_2
1.0	0.4

with the rule: IF *X* is A THEN *Y* is B

given fact A':

x_1	x_2	x_3
0.6	0.9	0.7

⇒

R	x_1	x_2	x_3
y_1	1.0	1.0	1.0
y_2	0.9	0.4	0.8

with Imp(a,b) = min{1, 1-a+b }

thus: A' ∘ R = B'

with max-min-composition

$$\begin{pmatrix} 0.6 & 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 0.9 \\ 1.0 & 0.4 \\ 1.0 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.7 \end{pmatrix}$$

■

inference from fuzzy statements

- conventional: modus trollens

$a \Rightarrow b$
$\overline{b}$
———
$\overline{a}$

- fuzzy: generalized modus trollens (GMT)

IF *X* is A, THEN *Y* is B
Y is B'
———
X is A'

e.g.: IF *heating* is hot, THEN *energy consumption* is high
energy consumption is normal
———
heating is warm

example: GMT

consider A:

x_1	x_2	x_3
0.5	1.0	0.6

B:

y_1	y_2
1.0	0.4

with the rule: IF *X* is A THEN *Y* is B

given fact B':

y_1	y_2
0.9	0.7

⇒

R	x_1	x_2	x_3
y_1	1.0	1.0	1.0
y_2	0.9	0.4	0.8

with Imp(a,b) = min{1, 1-a+b }

thus: B' ∘ R^{-1} = A'

with max-min-composition

$$\begin{pmatrix} 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 0.9 & 0.4 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.9 & 0.9 \end{pmatrix}$$

■

inference from fuzzy statements

- conventional: hypothetic syllogism

$a \Rightarrow b$
$b \Rightarrow c$
—————
$a \Rightarrow c$

- fuzzy: generalized HS

IF *X* is A, THEN *Y* is B
IF *Y* is B, THEN *Z* is C
—————
IF X is A, THEN Z is C

e.g.: IF *heating* is hot, THEN *energy consumption* is high
IF *energy consumption* is high, THEN *living* is expensive
—————
IF *heating* is hot, THEN *living* is expensive

example: GHS

let fuzzy sets A(x), B(x), C(x) be given

$\Rightarrow$ determine the three relations

$R_1(x,y) = Imp(A(x),B(y))$
$R_2(y,z) = Imp(B(y),C(z))$
$R_3(x,z) = Imp(A(x),C(z))$

and express them as matrices R_1, R_2, R_3

We say:
GHS is valid if $R_1 \circ R_2 = R_3$

So, ... what makes sense for Imp(·,·) ?

Imp(a,b) ought to express fuzzy version of implication ($a \Rightarrow b$)

conventional: $a \Rightarrow b$ identical to $\overline{a} \vee b$

But how can we calculate with fuzzy “boolean” expressions?

request: must be compatible to crisp version (and more) for $a,b \in \{ 0, 1 \}$

a	b	a ∧ b	t(a,b)
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

a	b	a ∨ b	s(a,b)
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

a	$\overline{a}$	c(a)
0	1	1
1	0	0

So, ... what makes sense for Imp(·,·) ?

1st approach: S implications

conventional: $a \Rightarrow b$ identical to $\overline{a} \vee b$

fuzzy: Imp(a, b) = s(c(a), b)

2nd approach: R implications

conventional: $a \Rightarrow b$ identcal to $\max\{ x \in \mathbb{B} : a \wedge x \leq b \}$

fuzzy: $Imp(a, b) = \max\{ x \in [0,1] : t(a, x) \leq b \}$

3rd approach: QL implications

conventional: $a \Rightarrow b$ identical to $\overline{a} \vee b \equiv \overline{a} \vee (a \wedge b)$ law of absorption

fuzzy: Imp(a, b) = s(c(a), t(a, b)) (dual tripel ?)

example: S implication Imp(a, b) = s(c_s(a), b) (c_s : std. complement)

1. Kleene-Dienes implication
   s(a, b) = max{ a, b } (standard) Imp(a,b) = max{ 1-a, b }
2. Reichenbach implication
   s(a, b) = a + b – ab (algebraic sum) Imp(a, b) = 1 – a + ab
3. Łukasiewicz implication
   s(a, b) = min{ 1, a + b } (bounded sum) Imp(a, b) = min{ 1, 1 – a + b }

example: R implicationen Imp(a, b) = max{ x ∈[0,1] : t(a, x) ≤ b }

1. Gödel implication
   t(a, b) = min{ a, b } (std.) Imp(a, b) = $\begin{cases} 1 & \text{, if } a \le b \\ b & \text{, else} \end{cases}$
2. Goguen implication
   t(a, b) = ab (algeb. product) Imp(a, b) = $\begin{cases} 1 & \text{, if } a \le b \\ \frac{b}{a} & \text{, else} \end{cases}$
3. Łukasiewicz implication
   t(a, b) = max{ 0, a + b – 1 } (bounded diff.) Imp(a, b) = min{ 1, 1 – a + b }

example: QL implication Imp(a, b) = s(c(a), t(a, b))

1. Zadeh implication
   t(a, b) = min { a, b } (std.) Imp(a, b) = max{ 1 – a, min{a, b} }
   s(a,b) = max{ a, b } (std.)
2. „NN“ implication ☺ (Klir/Yuan 1994)
   t(a, b) = ab (algebr. prd.) Imp(a, b) = 1 – a + a^2b
   s(a,b) = a + b – ab (algebr. sum)
3. Kleene-Dienes implication
   t(a, b) = max{ 0, a + b – 1 } (bounded diff.) Imp(a, b) = max{ 1-a, b }
   s(a,b) = min { 1, a + b) (bounded sum)

axioms for fuzzy implications

1. a ≤ b implies Imp(a, x) ≥ Imp(b, x) — monotone in 1st argument
2. a ≤ b implies Imp(x, a) ≤ Imp(x, b) — monotone in 2nd argument
3. Imp(0, a) = 1 — dominance of falseness
4. Imp(1, b) = b — neutrality of trueness
5. Imp(a, a) = 1 — identity
6. Imp(a, Imp(b, x)) = Imp(b, Imp(a, x)) — exchange property
7. Imp(a, b) = 1 iff a ≤ b — boundary condition
8. Imp(a, b) = Imp(c(b), c(a)) — contraposition
9. Imp(·, ·) is continuous — continuity

characterization of fuzzy implication

Theorem:
Imp: $[0,1] \times [0,1] \rightarrow [0,1]$ satisfies axioms 1-9 for fuzzy implications for a certain fuzzy complement $c(\cdot)$ $\Leftrightarrow$

$\exists$ strictly monotone increasing, continuous function $f: [0,1] \rightarrow [0, \infty)$ with

- $f(0) = 0$
- $\forall a, b \in [0,1]$: $Imp(a, b) = f^{(-1)}(f(1) – f(a) + f(b))$
- $\forall a \in [0,1]$: $c(a) = f^{-1}(f(1) – f(a))$

Proof: Smets & Magrez (1987). ■

examples: (in tutorial)

choosing an „appropriate" fuzzy implication ...

apt quotation: (Klir & Yuan 1995, p. 312)

„To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem."

guideline:
GMP, GMT, GHS should be compatible with MP, MT, HS
for fuzzy implication in calculations with relations:
$B(y) = \sup \{ t(A(x), Imp(A(x), B(y))) : x \in \mathcal{X} \}$

example:
Gödel implication for t-norm = bounded difference