

Winter Term 2011/12

Prof. Dr. Günter Rudolph Lehrstuhl für Algorithm Engineering (LS 11)

Fakultät für Informatik TU Dortmund

Fuzzy Sets

Considered so far: Standard fuzzy operators

• $A^{c}(x) = 1 - A(x)$

• $(A \cap B)(x) = \min \{ A(x), B(x) \}$ • $(A \cup B)(x) = \max \{ A(x), B(x) \}$

Defined via axioms.

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Creation via generators.

Lecture 06

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⇒ Compatible with operators for crisp sets with membership functions with values in $\mathbb{B} = \{0, 1\}$

∃ Non-standard operators? ⇒ Yes! Innumerable many!

Plan for Today

Fuzzy sets

Generators

Dual tripels

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Axioms of fuzzy complement, t- and s-norms

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involutive

ad (A3): ☑

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ad (A4): 1 - (1 - a) = a

monotone decreasing

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Definition

A function c: $[0,1] \rightarrow [0,1]$ is a *fuzzy complement* iff

Fuzzy Complement: Axioms

c(0) = 1 and c(1) = 0.

 \forall a, b \in [0,1]: a \leq b \Rightarrow c(a) \geq c(b).

"nice to have":

Examples:

a) standard fuzzy complement c(a) = 1 - a

ad (A1): c(0) = 1 - 0 = 1 and c(1) = 1 - 1 = 0

ad (A2): c'(a) = -1 < 0 (monotone decreasing)

 $c(\cdot)$ is continuous.

 $\forall \ a \in [0,1]: c(c(a)) = a$

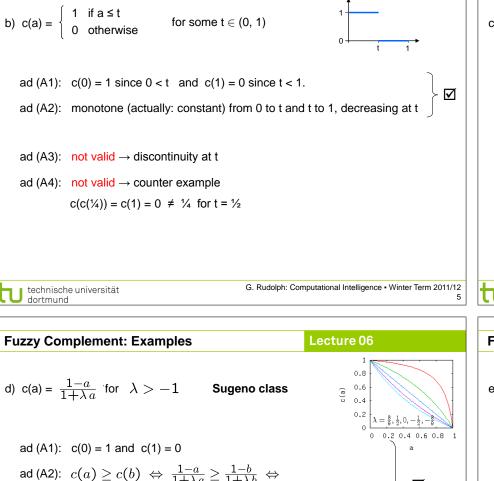
(A1)

(A2)

(A3)

(A4)

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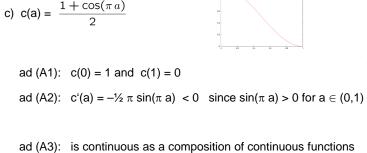


 $(1-a)(1+\lambda b) > (1-b)(1+\lambda a) \Leftrightarrow$

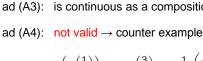
 $b(\lambda + 1) \ge a(\lambda + 1) \Leftrightarrow b \ge a$

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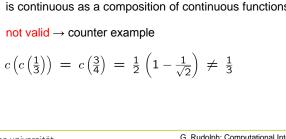
Fuzzy Complement: Examples

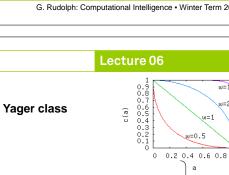


Fuzzy Complement: Examples



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ad (A2): $(1 - a^w)^{1/w} \ge (1 - b^w)^{1/w} \iff 1 - a^w \ge 1 - b^w \iff$

ad (A1): c(0) = 1 and c(1) = 0

 $a^w \le b^w \Leftrightarrow a \le b$

e)
$$c(a) = (1 - a^w)^{1/w}$$
 for $w > 0$

$$\left(1-\frac{1}{\sqrt{2}}\right)$$

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ad (A3): is continuous as a composition of continuous functions ad (A4): $c(c(a)) = c\left((1-a^w)^{\frac{1}{w}}\right) = \left(1-\left[(1-a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$

 $= (1 - (1 - a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$



ad (A3): is continuous as a composition of continuous functions

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ad (A4): $c(c(a)) = c\left(\frac{1-a}{1+\lambda a}\right) = \frac{1-\frac{1-a}{1+\lambda a}}{1+\lambda \frac{1-a}{1+\lambda}} = \frac{a(\lambda+1)}{\lambda+1} = a$

 $\overline{\mathbf{A}}$

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 $\overline{\mathbf{Q}}$

If function
$$c:[0,1] \to [0,1]$$
 satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a^* with $c(a^*) = a^*$.

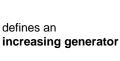
Proof:

one fixed point \to see example (a) \to intersection with bisectrix

no fixed point \to see example (b) \to no intersection with bisectrix

assume $\exists \ n > 1$ fixed points, for example a^* and b^* with $a^* < b^*$
 $\Rightarrow c(a^*) = a^*$ and $c(b^*) = b^*$ (fixed points)
 $\Rightarrow c(a^*) < c(b^*)$ with $a^* < b^*$ impossible if $c(\cdot)$ is monotone decreasing
 \Rightarrow contradiction to axiom (A2)

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g(-1)(x) pseudo-inverse

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Examples

a)
$$g(x) = x$$
 $\Rightarrow g^{-1}(x) = x$ $\Rightarrow c(a) = 1 - a$ (Standard)
b) $g(x) = x^w$ $\Rightarrow g^{-1}(x) = x^{1/w}$ $\Rightarrow c(a) = (1 - a^w)^{1/w}$ (Yager class, $w > 0$)

c)
$$g(x) = \log(x+1) \Rightarrow g^{-1}(x) = e^x - 1 \Rightarrow c(a) = \exp(\log(2) - \log(a+1)) - 1$$

= $\frac{1-a}{2}$ (Sugeno class. $\lambda = 1$)

If function c: $[0,1] \rightarrow [0,1]$ satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a^* with $c(a^*) = a^*$.

Proof:

Theorem

Intermediate value theorem →

Examples:

If $c(\cdot)$ continuous (A3) and $c(0) \ge c(1)$ (A1/A2)

then $\forall v \in [c(1), c(0)] = [0,1]$: $\exists a \in [0,1]$: c(a) = v.

⇒ there must be an intersection with bisectrix

Fuzzy Complement: Fixed Points

⇒ a fixed point exists and by previous theorem there are no other fixed points! ■

(a) c(a) = 1 - a $\Rightarrow a = 1 - a$ $\Rightarrow a^* = \frac{1}{2}$

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(b) $c(a) = (1 - a^w)^{1/w}$ $\Rightarrow a = (1 - a^w)^{1/w}$ $\Rightarrow a^* = (\frac{1}{2})^{1/w}$

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Fuzzy Complement: 1st Characterization

Examples

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d) $g(a) = \frac{1}{\lambda} \log_e(1 + \lambda a)$ for $\lambda > -1$

• $a(0) = \log_e(1) = 0$

• strictly monotone increasing since $g'(a) = \frac{1}{1+\lambda a} > 0$ for $a \in [0,1]$

 $c(a) = g^{-1} \left(\frac{\log(1+\lambda)}{\lambda} - \frac{\log(1+\lambda a)}{\lambda} \right)$

• inverse function on [0,1] is $g^{-1}(a) = \frac{\exp(\lambda a) - 1}{\lambda}$, thus

 $= \frac{\exp(\log(1+\lambda) - \log(1+\lambda a)) - 1}{\lambda}$ $= \frac{1}{\lambda} \left(\frac{1+\lambda}{1+\lambda a} - 1 \right) = \frac{1-a}{1+\lambda a}$ (Sugeno Complement)

• g(0) = 0

Theorem c: $[0,1] \rightarrow [0,1]$ is involutive fuzzy complement iff

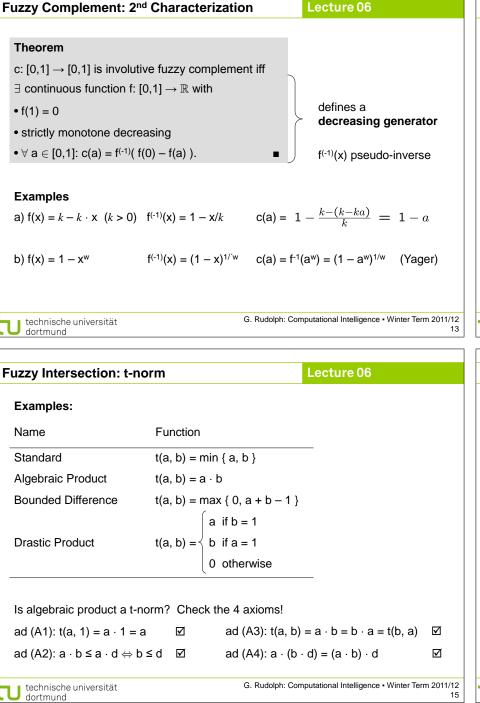
strictly monotone increasing

 \exists continuous function g: $[0,1] \rightarrow \mathbb{R}$ with

• \forall a \in [0,1]: c(a) = $g^{(-1)}(g(1) - g(a))$.

Fuzzy Complement: Fixed Points

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"nice to have" (A5) t(a, b) is continuous (continuity) (A6) t(a, a) < a(subidempotent)

A function $t:[0,1] \times [0,1] \rightarrow [0,1]$ is a *fuzzy intersection* or *t-norm* iff

(A7) $a_1 \le a_2$ and $b_1 \le b_2 \Rightarrow t(a_1, b_1) \le t(a_2, b_2)$ (strict monotonicity) **Note**: the only idempotent t-norm is the standard fuzzy intersection

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Fuzzy Intersection: Characterization Lecture 06

Theorem Function t: $[0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm \Leftrightarrow

Fuzzy Intersection: t-norm

(A2) $b \le d \Rightarrow t(a, b) \le t(a, d)$

(A4) t(a, t(b, d)) = t(t(a, b), d)

Definition

(A1) t(a, 1) = a

(A3) t(a,b) = t(b, a)

 \exists decreasing generator f:[0,1] $\rightarrow \mathbb{R}$ with t(a, b) = f⁽⁻¹⁾(f(a) + f(b)).

• $f'(x) = -1/x^2 < 0$ (monotone decreasing)

inverse function is $f^{-1}(x) = \frac{1}{x+1}$

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Example: f(x) = 1/x - 1 is decreasing generator since

• f(x) is continuous \checkmark • f(1) = 1/1 - 1 = 0

 \Rightarrow t(a, b) = $f^{-1} \left(\frac{1}{a} + \frac{1}{b} - 2 \right) = \frac{1}{\frac{1}{a} + \frac{1}{b} - 1} = \frac{ab}{a + b - ab}$

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 \square

 \square

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(monotonicity)

(commutative)

(associative)

Definition A function s: [0,1] × [0,1] → [0,1] is a fuzzy union or s-norm or t-conorm iff (A1) s(a, 0) = a (A2) b ≤ d ⇒ s(a, b) ≤ s(a, d) (monotonicity) (Algebraic Sum (A3) s(a, b) = s(b, a) (commutative) (Algebraic Sum (A4) s(a, s(b, d)) = s(s(a, b), d) (associative) Drastic Union "nice to have" (A5) s(a, b) is continuous (continuity) (A6) s(a, a) > a (superidempotent) (A7) a₁ ≤ a₂ and b₁ ≤ b₂ ⇒ s(a₁, b₁) ≤ s(a₂, b₂) (strict monotonicity) Note: the only idempotent s-norm is the standard fuzzy union Is algebraic sum a t-norn ad (A1): s(a, 0) = a + 0 - ad (A2): a + b - a · b ≤ ad (A2): a + b -	Fuzzy Union: s-norm	Lecture 06	Fuzzy Union: s-norm
A function s: $[0,1] \times [0,1] \to [0,1]$ is a <i>fuzzy union</i> or <i>s</i> -norm or <i>t</i> -conorm iff (A1) s(a, 0) = a (A2) b ≤ d ⇒ s(a, b) ≤ s(a, d) (monotonicity) (A3) s(a, b) = s(b, a) (commutative) (A4) s(a, s(b, d)) = s(s(a, b), d) (associative) "nice to have" (A5) s(a, b) is continuous (continuity) (A6) s(a, a) > a (superidempotent) (A7) $a_1 \le a_2$ and $b_1 \le b_2 \Rightarrow s(a_1, b_1) \le s(a_2, b_2)$ (strict monotonicity) Note: the only idempotent s-norm is the standard fuzzy union Fuzzy Union: Characterization Fuzzy Union: Characterization Example: g(x) = -log(1 - a) is decreasing generator since • g(x) = -log(1 - a) is decreasing generator since • g(x) = 1/(1-a) > 0 (monotonic increasing) inverse function is $g^{-1}(x) = 1 - \exp(-a)$ ⇒ $s(a, b) = g^{-1}(-\log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) + \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) + \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) + \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a) - \log(1 - a) - \log(1 - a))$ = $1 - \exp(\log(1 - a$	•		
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(A2) $b \le d \Rightarrow s(a, b) \le s(a, d)$ (monotonicity) (A3) $s(a, b) = s(b, a)$ (commutative) (A4) $s(a, s(b, d)) = s(s(a, b), d)$ (associative) "nice to have" (A5) $s(a, b)$ is continuous (A6) $s(a, a) > a$ (superidempotent) (A7) $a_1 \le a_2$ and $b_1 \le b_2 \Rightarrow s(a_1, b_1) \le s(a_2, b_2)$ (strict monotonicity) Note: the only idempotent s-norm is the standard fuzzy union Fuzzy Union: Characterization Lecture 06 Theorem Function $s: [0,1] \times [0,1] \rightarrow [0,1]$ is a s-norm $\Leftrightarrow \exists$ increasing generator $g: [0,1] \rightarrow \mathbb{R}$ with $s(a, b) = g^{(-1)}(g(a) + g(b))$. Example: $g(x) = -\log(1-a)$ is decreasing generator since $g(x) = -\log(1-a)$ is decreasing generator since $g(x) = -\log(1-a) > 0$ (monotone increasing) $g: s(a, b) = g^{-1}(-\log(1-a) - \log(1-b))$ inverse function is $g^{-1}(x) = 1 - \exp(\log(1-a) - \log(1-b))$ and $g: s(a, b) = a + b - a$	A function s:[0,1] \times [0,1] \rightarrow [0,1] is a <i>fuzzy uni</i>	ion or s-norm or t-conorm iff	Name
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Is algebraic sum a t-norm at (A1): $s(a, 0) = a + 0 - a$ at (A2): $a + b - a + b \le a$ Theorem Function s: $[0,1] \times [0,1] \to [0,1]$ is a s-norm \Leftrightarrow ∃ increasing generator g: $[0,1] \to \mathbb{R}$ with $s(a, b) = g^{(-1)}(g(a) + g(b))$. Example: $g(x) = -\log(1 - a)$ is decreasing generator since • $g(x)$ is continuous • $g(x) = -\log(1 - 0) = 0$ • $g'(x) = 1/(1 - a) > 0$ (monotone increasing) inverse function is $g^{-1}(x) = 1 - \exp(\log(1 - a) + \log(1 - b))$ $= 1 - \exp(\log(1 - a) + \log(1 -$			
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Fuzzy Union: Characterization Lecture 06 Theorem Function s: $[0,1] \times [0,1] \rightarrow [0,1]$ is a s-norm \Leftrightarrow ∃ increasing generator g: $[0,1] \rightarrow \mathbb{R}$ with s(a, b) = $g^{(-1)}(g(a) + g(b))$. Example: $g(x) = -\log(1-a)$ is decreasing generator since • $g(x)$ is continuous • $g(x) = \log(1-0) = 0$ • $g'(x) = 1/(1-a) > 0$ (monotone increasing) \boxdot inverse function is $g^{-1}(x) = 1 - \exp(-a)$ ⇒ $s(a, b) = g^{-1}(-\log(1-a) - \log(1-b))$ = $1 - \exp(\log(1-a) + \log(1-b))$ = $1 $	Note: the only idempotent s-norm is the standard fuzzy union		ad (A2): a + b − a · b ≤ a
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Drastic Union Is algebraic sum a t-norm? Check the 4 axioms! ad (A1): $s(a, 0) = a + 0 - a \cdot 0 = a$

ad (A3): ☑ ad (A2): $a + b - a \cdot b \le a + d - a \cdot d \Leftrightarrow b (1 - a) \le d (1 - a) \Leftrightarrow b \le d \square$ ad (A4): ☑ G. Rudolph: Computational Intelligence • Winter Term 2011/12

Lecture 06

Definition

s- and t-norm.

Let (c, s, t) be a tripel

If t and s are dual to c

then the tripel (c,s, t) is

called a dual tripel.

of fuzzy complement $c(\cdot)$,

Lecture 06

Combination of Fuzzy Operations

Function

 $s(a, b) = max \{ a, b \}$

 $s(a, b) = a + b - a \cdot b$ $s(a, b) = min \{ 1, a + b \}$

s(a, b) = b if a = 0

a if b = 0

1 otherwise

Background from classical set theory:

∩ and ∪ operations are dual w.r.t. complement since they obey DeMorgan's laws

• c(t(a, b)) = s(c(a), c(b))

A pair of t-norm $t(\cdot, \cdot)$ and s-norm $s(\cdot, \cdot)$ is said to be

dual with regard to the fuzzy complement c(·) iff

for all $a, b \in [0,1]$.

• c(s(a, b)) = t(c(a), c(b))

 $min \{ 1, a + b \}$

Examples of dual tripels s-norm complement 1 - amax { a, b } $a+b-a\cdot b$ 1 - a

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