Evolutionary Algorithms for Multiobjective Optimization

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Lecture "Introduction to Computational Intelligence"
Winter 2011/12
TU Dortmund, Dept. of Computer Science, LS11

25.01.2012

Multiobjective Optimization

Multiobjective Problem
\[ f : S \subseteq \mathbb{R}^n \rightarrow Z \subseteq \mathbb{R}^d, \min_{x \in \mathbb{R}^n} f(x) = (f_1(x), \ldots, f_d(x)) \]

How to relate vectors?

Pareto Dominance

Partial order among vectors in \( \mathbb{R}^d \) and thus in \( \mathbb{R}^n \)

\[ (1,1) \prec (5,5) \prec (8,8) \]
\[ (1,8) \parallel (5,5) \parallel (8,1) \]

\( a \preceq b \), \( a \) weakly dominates \( b \): \( \iff \forall i \in \{1, \ldots, d\} : a_i \leq b_i \)
\( a \prec b \), \( a \) dominates \( b \): \( \iff a \preceq b \) and \( a \neq b \), i.e., \( \exists i \in \{1, \ldots, d\} : a_i < b_i \)
\( a \parallel b \), \( a \) and \( b \) are incomparable: \( \iff \) neither \( a \preceq b \) nor \( b \preceq a \).

Real-world problems: various demands on problem solution
\Rightarrow multiple conflictive objective functions
**Aim of Optimization**

Pareto front: set of optimal solution vectors in $\mathbb{R}^d$, i.e.,
\[
\text{PF} = \{ x \in \mathbb{Z} \mid \nexists x' \in \mathbb{Z} \text{ with } x' \prec x \}
\]

Aim of optimization: find Pareto front?
PF maybe infinitively large
PF hard to hit exactly in continuous space
$\Rightarrow$ too ambitious!
Aim of optimization: approximate Pareto front!

**Scalarization**

Isn’t there an easier way?

Scalarize objectives to single-objective function:
\[
f : S \subseteq \mathbb{R}^n \rightarrow \mathbb{Z} \subseteq \mathbb{R}^2 \Rightarrow f_{\text{scal}} = w_1 f_1(x) + w_2 f_2(x)
\]
Result: single solution
Specify desired solution by choice of $w_1, w_2$

**Classification**

a-priori approach
first specify preferences, then optimize
more advanced scalarization techniques (e.g. Tschebyscheff)
allow to access all elements of PF
remaining difficulty:
how to express your desires through parameter values!?

a-posteriori approach
first optimize (approximate Pareto front), then choose solution
$\Rightarrow$ back to a-posteriori approach
$\Rightarrow$ state-of-the-art methods: evolutionary algorithms
Evolutionary Algorithms

Evolutionary Multiobjective Optimization Algorithms (EMOA)
Multiobjective Optimization Evolutionary Algorithms (MOEA)

- initialization
- evaluation of population
- variation (recombination/crossover, mutation)
- parent selection for reproduction
- evaluation of offspring
- selection of succeeding population
- termination condition
- fulfilled?
- stop
- evolution

Selection in EMOA

Selection requires sortable population to choose best individuals

How to sort d-dimensional objective vectors?

Primary selection criterion:
use Pareto dominance relation to sort comparable individuals

Secondary selection criterion:
apply additional measure to incomparable individuals to enforce order

What to change in case of multiobjective optimization?
Selection!
Remaining operators may work on search space only

Non-dominated Sorting

Example for primary selection criterion
partition population into sets of mutually incomparable solutions (antichains)

- non-dominated set: best elements of set
  \[ \text{NDS}(M) = \{ x \in M \mid \nexists x' \in M \text{ with } x' \prec x \} \]

Simple algorithm:
iteratively remove non-dominated set until population empty

Non-dominated Sorting

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NSGA-II

Popular EMOA: Non-dominated Sorting Genetic Algorithm II

\((\mu + \mu)\)-selection:
1. perform non-dominated sorting on all \(\mu + \mu\) individuals
2. take best subsets as long as they can be included completely
3. if population size \(\mu\) not reached but next subset does not fit in completely:
   apply secondary selection criterion *crowding distance* to that subset
4. fill up population with best ones w.r.t. the *crowding distance*

NSGA-II

Crowding distance:
- \(1/2\) perimeter of empty bounding box around point
- value of infinity for boundary points
- large values good

Difficulties of Selection

imagine point in the middle of the search space
\(d = 2\): 1/4 better, 1/4 worse, 1/2 incomparable
\(d = 3\): 1/8 better, 1/8 worse, 3/4 incomparable
general: fraction \(2^{-d+1}\) comparable, decreases exponentially

⇒ typical case: all individuals incomparable
⇒ mainly secondary selection criterion in operation

Drawback of crowding distance:
rewards spreading of points, does not reward approaching the Pareto front
⇒ NSGA-II diverges for large \(d\), difficulties already for \(d = 3\)
Difficulties of Selection

Secondary selection criterion has to be meaningful!

Desired: choose best subset of size $\mu$ from individuals

How to compare sets of partially incomparable points?

⇒ use quality indicators for sets

One approach for selection

⇒ for each point: determine contribution to quality value of set
⇒ sort points according to contribution

Hypervolumen (S-metric) as Quality Measure

dominated hypervolume:
size of dominated space bounded by reference point

\[
H(M, r) := \text{Leb} \left( \bigcup_{i=1}^{m} [v(i), r] \right)
\]

\[M = \{v(1), v(2), \ldots, v(m)\}\]

$r$ reference point

to be maximized

SMS(S-Metric Selection)-EMOA

State-of-the-art EMOA

$(\mu + 1)$-selection

1. non-dominated sorting
2. in case of incomparability: contributions to hypervolume of subset
SMS(S-Metric Selection)-EMOA
State-of-the-art EMOA
$(\mu + 1)$-selection
1 non-dominated sorting
2 in case of incomparability: contributions to hypervolume of subset
**SMS(S-Metric Selection)-EMOA**

State-of-the-art EMOA

\((\mu + 1)\)-selection

1. non-dominated sorting
2. in case of incomparability: contributions to hypervolume of subset

**Computational complexity of hypervolume**

Lower Bound

\(\Omega(m \log m)\)

Upper Bound

\(O\left(m^{d/2} \cdot 2^{O(\log^{*} m)}\right)\)

proof: hypervolume as special case of Klee’s measure problem

**Conclusions on EMOA**

NSGA-II

- only suitable in case of \(d=2\) objective functions
- otherwise no convergence to Pareto front

SMS-EMOA

- also effective for \(d > 2\) due to hypervolume
- hypervolume calculation time-consuming

\(\Rightarrow\) use approximation of hypervolume

Other state-of-the-art EMOA, e.g.

- MO-CMA-ES: CMA-ES + hypervolume selection
- \(\epsilon\)-MOEA: objective space partitioned into grid, only 1 point per cell
- MSOPS: selection acc. to ranks of different scalarizations

**Conclusions**

- real-world problems are often multiobjective
- Pareto dominance only a partial order
- a priori: parameterization difficult
- a posteriori: choose solution after knowing possible compromises
- state-of-the-art a posteriori methods: EMOA, MOEA
- EMOA require sortable population for selection
- use quality measures as secondary selection criterion
- hypervolume: excellent quality measure, but computationally intensive
- use state-of-the-art EMOA, other may fail completely