Fuzzy Relations

Definition

Fuzzy relation = fuzzy set over crisp cartesian product \( X_1 \times X_2 \times \ldots \times X_n \)

\[ R(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } (x_1, x_2, \ldots, x_n) \in R \\ 0 & \text{otherwise} \end{cases} \]

notice that cartesian product is a set!

\( \Rightarrow \) all set operations remain valid!

example: Let \( X = \{ \text{New York}, \text{Paris} \} \) and \( Y = \{ \text{Beijing}, \text{New York}, \text{Dortmund} \} \).

relation \( R = \text{“very far away”} \)

<table>
<thead>
<tr>
<th></th>
<th>New York</th>
<th>Paris</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beijing</td>
<td>1.0</td>
<td>0.9</td>
</tr>
<tr>
<td>New York</td>
<td>0.0</td>
<td>0.7</td>
</tr>
<tr>
<td>Dortmund</td>
<td>0.6</td>
<td>0.3</td>
</tr>
</tbody>
</table>

appropriate representation: n-dimensional membership matrix
Fuzzy Relations

Definition
Let $R(X, Y)$ be a fuzzy relation with membership matrix $R$. The inverse fuzzy relation to $R(X, Y)$, denoted $R^{-1}(Y, X)$, is a relation on $Y \times X$ with membership matrix $R^{-1} = R'$.

Remark: $R'$ is the transpose of membership matrix $R$.

Evidently: $(R^{-1})^{-1} = R$ since $(R')' = R$.

Definition
Let $P(X, Y)$ and $Q(Y, Z)$ be fuzzy relations. The operation $\circ$ on two relations, denoted $(P \circ Q)(x, z)$, is termed max-min-composition iff
$$R(x, z) = (P \circ Q)(x, z) = \max_{y \in Y} \min \{ P(x, y), Q(y, z) \}.$$ 

max-prod composition
$$(P \circ Q)(x, z) = \max_{y \in Y} \{ P(x, y) \cdot Q(y, z) \}.$$ 

generalization: sup-t composition
$$(P \circ Q)(x, z) = \sup_{y \in Y} \{ t(P(x, y), Q(y, z)) \}$$

where $t(\ldots)$ is a t-norm

Further methods for realizing compositions of relations:

• reflexive
  $\iff \forall x \in \mathcal{X}: R(x, x) = 1$

• irreflexive
  $\iff \exists x \in \mathcal{X}: R(x, x) < 1$

• antireflexive
  $\iff \forall x \in \mathcal{X}: R(x, x) < 1$

• symmetric
  $\iff \forall (x, y) \in \mathcal{X} \times \mathcal{X}: R(x, y) = R(y, x)$

• asymmetric
  $\iff \exists (x, y) \in \mathcal{X} \times \mathcal{X}: R(x, y) \neq R(y, x)$

• antisymmetric
  $\iff \forall (x, y) \in \mathcal{X} \times \mathcal{X}: R(x, y) \neq R(y, x)$

• transitive
  $\iff \forall (x, z) \in \mathcal{X} \times \mathcal{X}: R(x, z) \geq \max_{y \in \mathcal{Y}} \{ R(x, y), R(y, z) \}$

• intransitive
  $\iff \exists (x, z) \in \mathcal{X} \times \mathcal{X}: R(x, z) < \max_{y \in \mathcal{Y}} \{ R(x, y), R(y, z) \}$

• antitransitive
  $\iff \forall (x, z) \in \mathcal{X} \times \mathcal{X}: R(x, z) < \min_{y \in \mathcal{Y}} \{ R(x, y), R(y, z) \}$

Actually, here: max-min-transitivity ($\rightarrow$ in general: sup-t-transitivity)
binary fuzzy relation on $\mathcal{X} \times \mathcal{X}$: example

Let $\mathcal{X}$ be the set of all cities in Germany. Fuzzy relation $R$ is intended to represent the concept of „very close to“.

- $R(x,x) = 1$, since every city is certainly very close to itself. ⇒ reflexive
- $R(x,y) = R(y,x)$: if city $x$ is very close to city $y$, then also vice versa. ⇒ symmetric
- $R(\text{Dortmund, Essen}) = 0.8$
  $R(\text{Essen, Duisburg}) = 0.7$
  $R(\text{Dortmund, Duisburg}) = 0.5$
  $R(\text{Dortmund, Hagen}) = 0.9$
  ⇒ intransitive

linguistic variable:
variable that can attain several values of linguistic / verbal nature
E.g.: color can attain values red, green, blue, yellow, …

values (red, green, …) of linguistic variable are called linguistic terms

linguistic terms are associated with fuzzy sets

fuzzy proposition
$p$: temperature is high

- LV may be associated with several LT: high, medium, low, …
- high, medium, low temperature are fuzzy sets over numerical scale of crisp temperatures
- trueness of fuzzy proposition „temperature is high“ for a given concrete crisp temperature value $v$ is interpreted as equal to the degree of membership $\text{high}(v)$ of the fuzzy set high
**Fuzzy Logic**

**Lecture 07**

**fuzzy proposition**

\[ p: \text{V is F} \]

linguistic variable (LV) \hspace{1cm} linguistic term (LT)

actually:

\[ p: \text{V is } F(v) \]

and

\[ T(p) = F(v) \text{ for a concrete crisp value } v \]

trueeness(p)

---

**Fuzzy Logic**

**Lecture 07**

**fuzzy proposition**

\[ p: \text{IF X is A, THEN Y is B} \]

\[ \text{LV} \hspace{1cm} \text{LT} \hspace{1cm} \text{LV} \hspace{1cm} \text{LT} \]

How can we determine / express degree of trueness \( T(p) \)?

- For crisp, given values \( x, y \), we know \( A(x) \) and \( B(y) \)
- \( A(x) \) and \( B(y) \) must be processed to single value via relation \( R \)
- \( R( x, y ) = \text{function}( A(x), B(y) ) \) is fuzzy set over \( X \times Y \)
- as before: interprete \( T(p) \) as degree of membership \( R(x,y) \)

---

**Fuzzy Logic**

**Lecture 07**

**fuzzy proposition**

\[ p: \text{IF heating is hot, THEN energy consumption is high} \]

\[ \text{LV} \hspace{1cm} \text{LT} \hspace{1cm} \text{LV} \hspace{1cm} \text{LT} \]

expresses relation between

a) temperature of heating and  
b) quantity of energy consumption

\[ p: (\text{heating, energy consumption}) \in R \]

---

**Fuzzy Logic**

**Lecture 07**

**fuzzy proposition**

\[ p: \text{IF X is A, THEN Y is B} \]

\[ \text{A is fuzzy set over X} \]

\[ \text{B is fuzzy set over Y} \]

\[ \text{R is fuzzy set over } X \times Y \]

\[ \forall (x,y) \in X \times Y: \ R(x, y) = \text{Imp}( A(x), B(y) ) \]

What is \( \text{Imp}(\cdot, \cdot) \)?

\[ \Rightarrow \text{“appropriate” fuzzy implication } [0,1] \times [0,1] \rightarrow [0,1] \]
assumption: we know an “appropriate” \( \text{Imp}(a, b) \).
How can we determine the degree of trueness \( T(p) \) ?

example:
let \( \text{Imp}(a, b) = \min\{1, 1 - a + b\} \) and consider fuzzy sets
\[
A: \begin{array}{ccc}
  x_1 & x_2 & x_3 \\
  0.1 & 0.8 & 1.0 \\
\end{array}
\]
\[
B: \begin{array}{cc}
  y_1 & y_2 \\
  0.5 & 1.0 \\
\end{array}
\]

\[
\begin{array}{ccc}
R & x_1 & x_2 & x_3 \\
  y_1 & 1.0 & 0.7 & 0.5 \\
  y_2 & 1.0 & 1.0 & 1.0 \\
\end{array}
\]

\( \Rightarrow \) z.B.
\( R(x_2, y_1) = \text{Imp}(A(x_2), B(y_1)) = \text{Imp}(0.8, 0.5) = \min\{1.0, 0.7\} = 0.7 \)
and \( T(p) \) for \((x_2, y_1)\) is \( R(x_2, y_1) = 0.7 \)

inference from fuzzy statements

- Let relationship between \( x \) and \( y \) be a relation \( R \) on \( X \times Y \)
  - \( \text{IF } x = x \text{ THEN } y \in B = \{ y \in Y : (x, y) \in R \} \)
  - \( \text{IF } x \in A \text{ THEN } y \in B = \{ y \in Y : (x, y) \in R, \; x \in A \} \)

Also expressible via characteristic functions of sets \( A, B, R \):
\[
\begin{array}{c}
\forall y \in Y: B(y) = \sup_{x \in X} \min\{ A(x), R(x, y) \}
\end{array}
\]

Composition rule of inference (in matrix form):
\( B' = A' \circ R \)
**Fuzzy Logic**

### Lecture 07

#### Inference from fuzzy statements

- **Conventional:**
  - **Modus Ponens:**
    \[
    a \implies b \quad \frac{a}{b}
    \]

- **Fuzzy:**
  - **Generalized Modus Ponens (GMP):**
    \[
    \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \quad \frac{X \text{ is } A'}{Y \text{ is } B'}
    \]

  **Example:**
  - **Heating is hot, THEN Energy consumption is high**
  - **Heating is warm, Energy consumption is normal**

**Example:**

**GMP**

Consider with the rule: IF \( X \) is \( A \) THEN \( Y \) is \( B \)

\[
\begin{array}{ccc}
a & x_1 & x_2 & x_3 \\
0.5 & 1.0 & 0.6 \\
\end{array}
\]

\[
\begin{array}{ccc}
b & y_1 & y_2 \\
1.0 & 0.4 \\
\end{array}
\]

With the rule: IF \( X \) is \( A \) THEN \( Y \) is \( B \)

**Given Fact:**

\[
\begin{array}{ccc}
A' & x_1 & x_2 & x_3 \\
0.6 & 0.9 & 0.7 \\
\end{array}
\]

\[
\begin{array}{ccc}
R & x_1 & x_2 & x_3 \\
y_1 & 1.0 & 1.0 & 1.0 \\
y_2 & 0.9 & 0.4 & 0.8 \\
\end{array}
\]

With \( \text{Imp}(a,b) = \min\{1, 1-a+b\} \)

Thus:

\[
A' \circ R = B'
\]

\[
\left( 0.6 \ 0.9 \ 0.7 \right) \circ \left( \begin{array}{ccc}
1.0 & 0.9 \\
1.0 & 0.4 \\
1.0 & 0.8 \\
\end{array} \right) = \left( 0.9 \ 0.7 \right)
\]

**Example:**

**GMT**

Consider with the rule: IF \( X \) is \( A \) THEN \( Y \) is \( B \)

\[
\begin{array}{ccc}
a & x_1 & x_2 & x_3 \\
0.5 & 1.0 & 0.6 \\
\end{array}
\]

\[
\begin{array}{ccc}
b & y_1 & y_2 \\
1.0 & 0.4 \\
\end{array}
\]

With the rule: IF \( X \) is \( A \) THEN \( Y \) is \( B \)

**Given Fact:**

\[
\begin{array}{ccc}
B' & y_1 & y_2 \\
0.9 & 0.7 \\
\end{array}
\]

\[
\begin{array}{ccc}
R & x_1 & x_2 & x_3 \\
y_1 & 1.0 & 1.0 & 1.0 \\
y_2 & 0.9 & 0.4 & 0.8 \\
\end{array}
\]

With \( \text{Imp}(a,b) = \min\{1, 1-a+b\} \)

Thus:

\[
B' \circ R^{-1} = A'
\]

\[
\left( 0.9 \ 0.7 \right) \circ \left( \begin{array}{ccc}
1.0 & 1.0 & 1.0 \\
0.9 & 0.4 & 0.8 \\
\end{array} \right) = \left( 0.9 \ 0.9 \ 0.9 \right)
\]
inference from fuzzy statements

- conventional:
  hypothetic syllogism
  
  \[
  \begin{align*}
  a & \Rightarrow b \\
  b & \Rightarrow c \\
  a & \Rightarrow c
  \end{align*}
  \]

- fuzzy:
  generalized HS
  
  IF \( X \) is A, THEN \( Y \) is B
  IF \( Y \) is B, THEN \( Z \) is C
  IF \( X \) is A, THEN \( Z \) is C

  e.g.: IF heating is hot, THEN energy consumption is high
  IF energy consumption is high, THEN living is expensive
  IF heating is hot, THEN living is expensive

example: GHS

Let fuzzy sets \( A(x), B(x), C(x) \) be given

\[
\Rightarrow \text{determine the three relations}
\]

\[
\begin{align*}
R_1(x,y) & = \text{Imp}(A(x),B(y)) \\
R_2(y,z) & = \text{Imp}(B(y),C(z)) \\
R_3(x,z) & = \text{Imp}(A(x),C(z))
\end{align*}
\]

and express them as matrices \( R_1, R_2, R_3 \)

**We say:**
GHS is valid if \( R_1 \circ R_2 = R_3 \)

So, ... what makes sense for \( \text{Imp}(.,. \) ?

\( \text{Imp}(a,b) \) ought to express fuzzy version of implication \( (a \Rightarrow b) \)

conventional: \( a \Rightarrow b \) identical to \( \overline{a} \lor b \)

But how can we calculate with fuzzy “boolean” expressions?

**request:** must be compatible to crisp version (and more) for \( a, b \in \{0, 1\} \)

\[
\begin{array}{|c|c|c|c|}
\hline
a & b & a \land b & t(a,b) \\
\hline
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
a & b & a \lor b & s(a,b) \\
\hline
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & \overline{a} & c(a) \\
\hline
0 & 1 & 1 \\
1 & 0 & 0 \\
\hline
\end{array}
\]

**1st approach:** \( S \) implications

conventional: \( a \Rightarrow b \) identical to \( \overline{a} \lor b \)

fuzzy: \( \text{Imp}(a,b) = s(c(a),b) \)

**2nd approach:** \( R \) implications

conventional: \( a \Rightarrow b \) identical to \[ \max\{ x \in B : a \land x \leq b \} \]

fuzzy: \( \text{Imp}(a,b) = \max\{ x \in [0,1] : t(a,x) \leq b \} \)

**3rd approach:** \( QL \) implications

conventional: \( a \Rightarrow b \) identical to \[ \overline{a} \lor b \equiv \overline{a} \lor (a \land b) \] law of absorption

fuzzy: \( \text{Imp}(a,b) = s(c(a),t(a,b)) \) (dual triple?)
### Example: S Implication

\[ \text{Imp}(a, b) = s(c_s(a), b) \quad (c_s: \text{std. complement}) \]

1. **Kleene-Dienes Implication**
   
   \[ s(a, b) = \max\{ a, b \} \quad \text{(standard)} \]
   
   \[ \text{Imp}(a, b) = \max\{ 1-a, b \} \]

2. **Reichenbach Implication**
   
   \[ s(a, b) = a + b - ab \quad \text{(algebraic sum)} \]
   
   \[ \text{Imp}(a, b) = 1 - a + ab \]

3. **Łukasiewicz Implication**
   
   \[ s(a, b) = \min\{ 1, a + b \} \quad \text{(bounded sum)} \]
   
   \[ \text{Imp}(a, b) = \min\{ 1, 1 - a + b \} \]

### Example: R Implication

\[ \text{Imp}(a, b) = \max\{ x \in [0,1] : t(a, x) \leq b \} \]

1. **Gödel Implication**
   
   \[ t(a, b) = \min\{ a, b \} \quad \text{(std.)} \]
   
   \[ \text{Imp}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{else} \end{cases} \]

2. **Goguen Implication**
   
   \[ t(a, b) = ab \quad \text{(algeb. product)} \]
   
   \[ \text{Imp}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ a, & \text{else} \end{cases} \]

3. **Łukasiewicz Implication**
   
   \[ t(a, b) = \max\{ 0, a + b - 1 \} \quad \text{(bounded diff.)} \]
   
   \[ \text{Imp}(a, b) = \min\{ 1, 1 - a + b \} \]

### Example: QL Implication

\[ \text{Imp}(a, b) = s(c(a), t(a, b)) \]

1. **Zadeh Implication**
   
   \[ t(a, b) = \min\{ a, b \} \quad \text{(std.)} \]
   
   \[ \text{Imp}(a, b) = \max\{ 1-a, \min\{a, b\} \} \]

2. **“NN” Implication** (Klir/Yuan 1994)
   
   \[ t(a, b) = ab \quad \text{(algeb. prd.)} \]
   
   \[ \text{Imp}(a, b) = 1 - a + a^2b \]

3. **Kleene-Dienes Implication**
   
   \[ t(a, b) = \max\{ 0, a + b - 1 \} \quad \text{(bounded diff.)} \]
   
   \[ \text{Imp}(a, b) = \max\{ 1-a, b \} \]

\[ s(a, b) = \min\{ 1, a + b \} \quad \text{(bounded sum)} \]

### Axioms for Fuzzy Implications

1. \( a \leq b \) implies \( \text{Imp}(a, x) \geq \text{Imp}(b, x) \) monotone in 1st argument
2. \( a \leq b \) implies \( \text{Imp}(x, a) \leq \text{Imp}(x, b) \) monotone in 2nd argument
3. \( \text{Imp}(0, a) = 1 \) dominance of falseness
4. \( \text{Imp}(1, b) = b \) neutrality of trueness
5. \( \text{Imp}(a, a) = 1 \) identity
6. \( \text{Imp}(a, \text{Imp}(b, x)) = \text{Imp}(\text{Imp}(a, b), x) \) exchange property
7. \( \text{Imp}(a, \text{Imp}(b, x)) = \text{Imp}(\text{Imp}(a, b), x) \) boundary condition
8. \( \text{Imp}(a, b) = \text{Imp}(\text{Imp}(c(b), c(a)) \text{ contraposition}
9. \( \text{Imp}(\cdot, \cdot) \) is continuous continuity
characterization of fuzzy implication

Theorem:
Imp: [0,1] × [0,1] → [0,1] satisfies axioms 1-9 for fuzzy implications
for a certain fuzzy complement c(·) ⇔

∃ strictly monotone increasing, continuous function f: [0,1] → [0, ∞) with

- f(0) = 0
- ∀a, b ∈ [0,1]: Imp(a, b) = f⁻¹(f(1) – f(a) + f(b))
- ∀a ∈ [0,1]: c(a) = f⁻¹(f(1) – f(a))


examples: (in tutorial)

choosing an „appropriate“ fuzzy implication ...

apt quotation: (Klir & Yuan 1995, p. 312)
„To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem.“

guideline:
GMP, GMT, GHS should be compatible with MP, MT, HS
for fuzzy implication in calculations with relations:
B(y) = sup { t( A(x), Imp( A(x), B(y) ) ) : x ∈ X' }

eexample:
Gödel implication for t-norm = bounded difference