Computational Intelligence

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Plan for Today

- Fuzzy relations
- Fuzzy logic
  - Linguistic variables and terms
  - Inference from fuzzy statements

Fuzzy Relations

relations with conventional sets $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$:

$$R(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n) \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$$

notice that cartesian product is a **set**!
$\Rightarrow$ all set operations remain valid!

crisp membership function (of x to relation R)

$$R(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } (x_1, x_2, \ldots, x_n) \in R \\ 0 & \text{otherwise} \end{cases}$$

Fuzzy Relations

Definition

Fuzzy relation = fuzzy set over crisp cartesian product $\mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$ ■

→ each tuple $(x_1, \ldots, x_n)$ has a degree of membership to relation

→ degree of membership expresses *strength of relationship* between elements of tuple

appropriate representation: n-dimensional membership matrix

example: Let X = { New York, Paris } and Y = { Bejing, New York, Dortmund }.

relation R = "very far away"

membership matrix →

relation R	New York	Paris
Bejing	1.0	0.9
New York	0.0	0.7
Dortmund	0.6	0.3

Definition

Let R(X, Y) be a fuzzy relation with membership matrix $\mathbf{R}$. The ***inverse fuzzy relation*** to R(X,Y), denoted $R^{-1}(Y, X)$, is a relation on Y x X with membership matrix $\mathbf{R}'$. ■

Remark: $\mathbf{R}'$ is the transpose of membership matrix $\mathbf{R}$.

Evidently: $(R^{-1})^{-1} = R$ since $(\mathbf{R}')' = \mathbf{R}$

Definition

Let P(X, Y) and Q(Y, Z) be fuzzy relations. The operation ∘ on two relations, denoted P(X, Y) ∘ Q(Y, Z), is termed ***max-min-composition*** iff

$$R(x, z) = (P \circ Q)(x, z) = \max_{y \in Y} \min \{ P(x, y), Q(y, z) \}.$$ ■

Theorem

a) max-min composition on relations is associative.

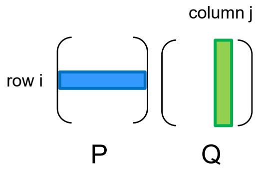
b) max-min composition on relations is not commutative.

c) $(P(X,Y) \circ Q(Y,Z))^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y, X)$.

membership matrix of max-min composition
determinable via “fuzzy matrix multiplication”: R = P ∘ Q

fuzzy matrix multiplication $r_{ij} = \max_k \min\{p_{ik}, q_{kj}\}$

crisp matrix multiplication $r_{ij} = \sum_k p_{ik} \cdot q_{kj}$



further methods for realizing compositions of relations:

max-prod composition

$$(P \odot Q)(x, z) = \max_{y \in \mathcal{Y}} \{P(x, y) \cdot Q(y, z)\}$$

<u>generalization:</u> sup-t composition

$$(P \circ Q)(x, z) = \sup_{y \in \mathcal{Y}} \{t(P(x, y), Q(y, z))\},$$ where t(.,.) is a t-norm

e.g.: t(a,b) = min{a, b} ⇒ max-min-composition

t(a,b) = a · b ⇒ max-prod-composition

Binary fuzzy relations on X x X : properties

- **reflexive** ⇔ ∀ x ∈ X : R(x,x) = 1
- **irreflexive** ⇔ ∃ x ∈ X : R(x,x) < 1
- **antireflexive** ⇔ ∀ x ∈ X : R(x,x) < 1

- **symmetric** ⇔ ∀ (x,y) ∈ X x X : R(x,y) = R(y,x)
- **asymmetric** ⇔ ∃ (x,y) ∈ X x X : R(x,y) ≠ R(y,x)
- **antisymmetric** ⇔ ∀ (x,y) ∈ X x X : R(x,y) ≠ R(y,x)

- **transitive** ⇔ ∀ (x,z) ∈ X x X : $R(x,z) \geq \max_{y \in X} \min \{ R(x,y), R(y,z) \}$
- **intransitive** ⇔ ∃ (x,z) ∈ X x X : $R(x,z) < \max_{y \in X} \min \{ R(x,y), R(y,z) \}$
- **antitransitive** ⇔ ∀ (x,z) ∈ X x X : $R(x,z) < \max_{y \in X} \min \{ R(x,y), R(y,z) \}$

actually, here: max-min-transitivity (→ in general: sup-t-transitivity)

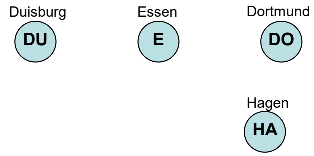
binary fuzzy relation on X x X: example

Let **X** be a subset of all cities in Germany.

Fuzzy relation R is intended to represent the concept of „very close to".

- R(x,x) = 1, since every city is certainly very close to itself.
  ⇒ **reflexive**
- R(x,y) = R(y,x): if city x is very close to city y, then also vice versa.
  ⇒ **symmetric**
-

R	DU	E	DO	HA
DU	1	0.7	0.5	0.4
E	0.7	1	0.8	0.8
DO	0.5	0.8	1	0.9
HA	0.4	0.8	0.9	1



$R(DO,DU) = 0.5 < \max_y \min\{R(DO,y), R(y, DU)\} = 0.7$

$R(E,DO) = 0.8 \geq \max_y \min\{R(E,y), R(y, DO)\} = 0.8$

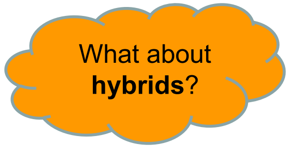
⇒ **intransitive**

crisp:
relation R is equivalence relation , R reflexive, symmetric, transitive

fuzzy:
relation R is similarity relation , R reflexive, symmetric, (max-min-) transitive

examples:

- *equivalence relation*: farm animals
  cattle, pigs, chicken, …
  R(cow, ox) = 1 but R(cow, hen) = 0



- *similarity relation*: farm animals
  cattle, pigs, chicken, horse, donkey, …
  R(mule, (male) donkey) = 0.5 and R(mule, (female) horse) = 0.5

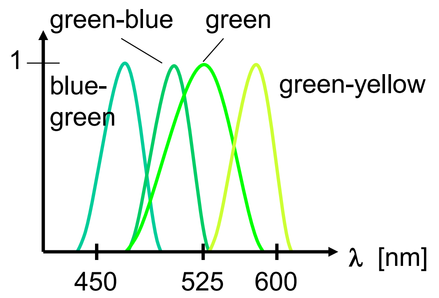
linguistic variable:

variable that can attain several values of lingustic / verbal nature

e.g.: **color** can attain values **red**, **green**, **blue**, **yellow**, …

values (red, green, …) of linguistic variable are called **linguistic terms**

linguistic terms are associated with fuzzy sets



fuzzy proposition

p: *temperature* is *high*

linguistic variable (LV) — linguistic term (LT)

- LV may be associated with several LT : *high, medium, low, …*
- *high, medium, low* temperature are fuzzy sets
  over numerical scale of crisp temperatures
- trueness of fuzzy proposition „temperature is high"
  for a given **concrete crisp** temperature value v
  is interpreted as equal to the degree of membership *high*(v)
  of the fuzzy set *high*

fuzzy proposition

p: *V* is F

↙ linguistic variable (LV)

↘ linguistic term (LT)

actually:

p: *V* is F(v)

and

T(p) = F(v) for a concrete crisp value v

↘ trueness(p)

establishes connection between *degree of membership* of a fuzzy set and the *degree of trueness* of a fuzzy proposition

fuzzy proposition

p: IF *heating* is *hot,* THEN *energy consumption* is *high*

(*heating*: LV, *hot*: LT, *energy consumption*: LV, *high*: LT)

expresses <u>relation</u> between

a) temperature of heating and

b) quantity of energy consumption

p: (*heating, energy consumption*) ∈ R ← relation

fuzzy proposition

p: IF *X* is A, THEN *Y* is B

(*X*: LV, A: LT, *Y*: LV, B: LT)

How can we determine / express degree of trueness T(p) ?

- For crisp, given values x, y we know A(x) and B(y)
- A(x) and B(y) must be processed to single value via relation R
- R(x, y) = function(A(x), B(y)) is fuzzy set over X x Y
- as before: interprete T(p) as degree of membership R(x,y)

fuzzy proposition

p: IF *X* is A, THEN *Y* is B

A is fuzzy set over X

B is fuzzy set over Y

R is fuzzy set over X x Y

$\forall$ (x,y) ∈ X x Y: R(x, y) = Imp(A(x), B(y))

What is Imp(·,·) ?

⇒ „appropriate“ fuzzy implication [0,1] x [0,1] → [0,1]

assumption: we know an „appropriate" Imp(a,b).

How can we determine the *degree of trueness* T(p) ?

example: (discrete case)

let Imp(a, b) = min{ 1, 1 – a + b } and consider fuzzy sets

A:

x_1	x_2	x_3
0.1	0.8	1.0

B:

y_1	y_2
0.5	1.0

⇒

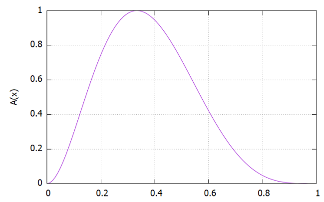
R	x_1	x_2	x_3
y_1	1.0	0.7	0.5
y_2	1.0	1.0	1.0

z.B.
$R(x_2, y_1) = Imp(A(x_2), B(y_1)) = Imp(0.8, 0.5) =$
$\min\{1.0, 0.7\} = 0.7$

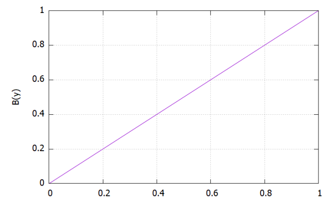
and T(p) for (x_2, y_1) is $R(x_2, y_1) = 0.7$ ■

example: (continuous case)

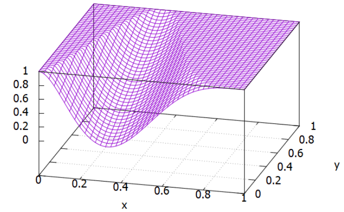
let Imp(a, b) = min{ 1, 1 – a + b } and consider fuzzy sets



$A(x) = \frac{729}{16}\, x^2\, (1-x)^4$ for $x \in [0,1]$



$B(y) = y$ for $y \in [0,1]$



$\Rightarrow\; R(x,y) = \min\{1, 1 - A(x) + B(y)\} = \min\{1, 1 - \frac{789}{16}\, x^2\, (1-x)^4 + y\}$

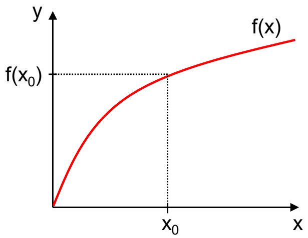
toward inference from fuzzy statements:

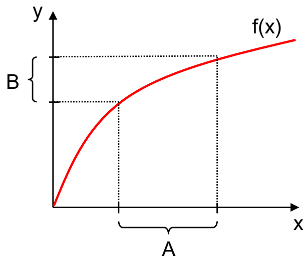
- let R = { (x, y) : y = f(x) } for a function f: $\mathbb{R} \to \mathbb{R}$

  IF X = { x_0 } THEN Y = { $f(x_0)$ }

- IF X ∈ A THEN Y ∈ B = { y ∈ $\mathcal{Y}$: y = f(x), x ∈ A }

crisp case: functional relationship





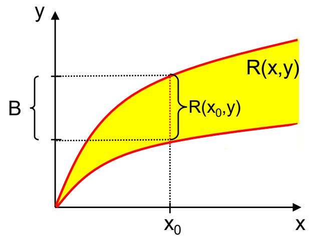
toward inference from fuzzy statements:

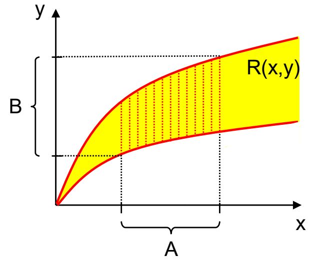
- let relationship between x and y be a relation R on $\mathcal{X} \times \mathcal{Y}$

  IF X = x_0 THEN Y ∈ B = { y ∈ $\mathcal{Y}$: (x_0, y) ∈ R }

- IF X ∈ A THEN Y ∈ B = { y ∈ $\mathcal{Y}$: (x, y) ∈ R, x ∈ A }

crisp case: relational relationship





toward inference from fuzzy statements:

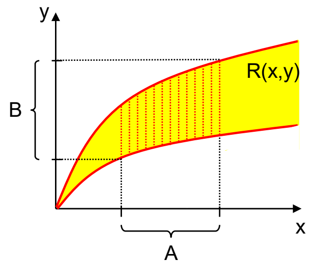
IF $X \in A$ THEN $Y \in B = \{ y \in \mathcal{Y}: (x, y) \in R,\ x \in A \}$

also expressible via characteristic functions of sets A, B, R:

$B(y) = 1$ iff $\exists x$: $A(x) = 1$ <u>and</u> $R(x, y) = 1$

$\Leftrightarrow \exists x: \min\{ A(x), R(x, y) \} = 1$

$\Leftrightarrow \max_{x \in \mathcal{X}} \min\{ A(x), R(x, y) \} = 1$



$$\forall y \in \mathcal{Y}: B(y) = \max_{x \in \mathcal{X}} \min \{ A(x), R(x, y) \}$$

inference from fuzzy statements

Now: A‘, B‘ fuzzy sets over $\mathcal{X}$ resp. $\mathcal{Y}$

Note:
A‘(x) is **not** the derivative of A(x)!
It is the membership function of fuzzy set A‘.

Assume: R(x,y) and A‘(x) are given.

<u>Idea:</u> Generalize characteristic function of B(y) to membership function B‘(y)

$\forall y \in \mathcal{Y}: B(y) = \max_{x \in \mathcal{X}} \min \{ A(x), R(x, y) \}$ characteristic functions

$\downarrow$

$\forall y \in \mathcal{Y}: B'(y) = \sup_{x \in \mathcal{X}} \min \{ A'(x), R(x, y) \}$ membership functions

composition rule of inference (in matrix form): $\mathbf{B^T = A \circ R}$

inference from fuzzy statements

- conventional: modus ponens

$a \Rightarrow b$
a
———
b

- fuzzy: generalized modus ponens (GMP)

IF *X* is A, THEN *Y* is B
X is A‘
———
Y is B‘

e.g.: IF *heating* is hot, THEN *energy consumption* is high
heating is warm
———
energy consumption is normal

example: GMP

consider

A:

x_1	x_2	x_3
0.5	1.0	0.6

B:

y_1	y_2
1.0	0.4

with the rule: IF *X* is A THEN *Y* is B

given fact A‘:

x_1	x_2	x_3
0.6	0.9	0.7

$\Rightarrow$

R	x_1	x_2	x_3
y_1	1.0	1.0	1.0
y_2	0.9	0.4	0.8

with Imp(a,b) = min{1, 1-a+b }

thus: $A' \circ R = B'$

with max-min-composition

$$\begin{pmatrix} 0.6 & 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 0.9 \\ 1.0 & 0.4 \\ 1.0 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.7 \end{pmatrix}$$

■

inference from fuzzy statements

- conventional:
  modus tollens

$$\frac{a \Rightarrow b \qquad \bar{b}}{\bar{a}}$$

- fuzzy:
  generalized modus tollens (GMT)

IF *X* is A, THEN *Y* is B
Y is B'

X is A'

e.g.:

IF *heating* is hot, THEN *energy consumption* is high
energy consumption is normal

---

heating is warm

example: GMT

consider

A:

x_1	x_2	x_3
0.5	1.0	0.6

B:

y_1	y_2
1.0	0.4

with the rule: IF *X* is A THEN *Y* is B

given fact B':

y_1	y_2
0.9	0.7

⇒

R	x_1	x_2	x_3
y_1	1.0	1.0	1.0
y_2	0.9	0.4	0.8

with Imp(a,b) = min{1, 1-a+b }

thus: $B' \circ R^{-1} = A'$

$$\begin{pmatrix} 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 0.9 & 0.4 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.9 & 0.9 \end{pmatrix}$$

with max-min-composition

■

inference from fuzzy statements

- conventional:
  hypothetic syllogism

$$\frac{a \Rightarrow b \qquad b \Rightarrow c}{a \Rightarrow c}$$

- fuzzy:
  generalized HS

IF *X* is A, THEN *Y* is B
IF *Y* is B, THEN *Z* is C

IF X is A, THEN Z is C

e.g.:

IF *heating* is hot, THEN *energy consumption* is high
IF *energy consumption* is high, THEN *living* is expensive

IF *heating* is hot, THEN *living* is expensive

example: GHS

let fuzzy sets A(x), B(x), C(x) be given

⇒ determine the three relations

$R_1(x,y) = Imp(A(x),B(y))$
$R_2(y,z) = Imp(B(y),C(z))$
$R_3(x,z) = Imp(A(x),C(z))$

and express them as matrices R_1, R_2, R_3

We say:

GHS is valid if $R_1 \circ R_2 = R_3$

So, ... what makes sense for Imp(·,·) ?

Imp(a,b) ought to express fuzzy version of implication ($a \Rightarrow b$)

conventional: $a \Rightarrow b$ identical to $\overline{a} \vee b$

But how can we calculate with fuzzy "boolean" expressions?

request: must be compatible to crisp version (and more) for $a,b \in \{0, 1\}$

a	b	$a \wedge b$	t(a,b)
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

a	b	$a \vee b$	s(a,b)
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

a	$\overline{a}$	c(a)
0	1	1
1	0	0

So, ... what makes sense for Imp(·,·) ?

1st approach: S implications

conventional: $a \Rightarrow b$ identical to $\overline{a} \vee b$

fuzzy: Imp(a, b) = s(c(a), b)

2nd approach: R implications

conventional: $a \Rightarrow b$ identical to $\max\{ x \in \{0,1\} : a \wedge x \le b \}$

fuzzy: $\text{Imp}(a, b) = \max\{ x \in [0,1] : t(a, x) \le b \}$

3rd approach: QL implications

conventional: $a \Rightarrow b$ identical to $\overline{a} \vee b \equiv \overline{a} \vee (a \wedge b)$ law of absorption

fuzzy: Imp(a, b) = s(c(a), t(a, b)) (dual tripel ?)

example: S implication $\text{Imp}(a, b) = s(c_s(a), b)$ (c_s : std. complement)

1. Kleene-Dienes implication

   s(a, b) = max{ a, b } (standard) Imp(a,b) = max{ 1-a, b }

2. Reichenbach implication

   s(a, b) = a + b – ab (algebraic sum) Imp(a, b) = 1 – a + ab

3. Łukasiewicz implication

   s(a, b) = min{ 1, a + b } (bounded sum) Imp(a, b) = min{ 1, 1 – a + b }

example: R implicationen $\text{Imp}(a, b) = \max\{ x \in [0,1] : t(a, x) \le b \}$

1. Gödel implication

   t(a, b) = min{ a, b } (std.) $\text{Imp}(a, b) = \begin{cases} 1 & \text{, if } a \le b \\ b & \text{, else} \end{cases}$

2. Goguen implication

   t(a, b) = ab (algeb. product) $\text{Imp}(a, b) = \begin{cases} 1 & \text{, if } a \le b \\ \frac{b}{a} & \text{, else} \end{cases}$

3. Łukasiewicz implication

   t(a, b) = max{ 0, a + b – 1 } (bounded diff.) Imp(a, b) = min{ 1, 1 – a + b }

example: QL implication $Imp(a, b) = s(c(a), t(a, b))$

1. Zadeh implication

   $t(a, b) = \min \{ a, b \}$ (std.) $Imp(a, b) = \max\{ 1 – a, \min\{a, b\} \}$
   $s(a,b) = \max\{ a, b \}$ (std.)

2. „NN" implication ☺ (Klir/Yuan 1994)

   $t(a, b) = ab$ (algebr. prd.) $Imp(a, b) = 1 – a + a^2b$
   $s(a,b) = a + b – ab$ (algebr. sum)

3. Kleene-Dienes implication

   $t(a, b) = \max\{ 0, a + b – 1 \}$ (bounded diff.) $Imp(a, b) = \max\{ 1-a, b \}$
   $s(a,b) = \min \{ 1, a + b)$ (bounded sum)

axioms for fuzzy implications

1. $a \le b$ implies $Imp(a, x) \ge Imp(b, x)$	monotone in 1st argument
2. $a \le b$ implies $Imp(x, a) \le Imp(x, b)$	monotone in 2nd argument
3. $Imp(0, a) = 1$	dominance of falseness
4. $Imp(1, b) = b$	neutrality of trueness
5. $Imp(a, a) = 1$	identity
6. $Imp(a, Imp(b, x)) = Imp(b, Imp(a, x))$	exchange property
7. $Imp(a, b) = 1$ iff $a \le b$	boundary condition
8. $Imp(a, b) = Imp(c(b), c(a))$	contraposition
9. $Imp(\cdot,\cdot)$ is continuous	continuity

Caution!

Not all S-, R-, QL- implications obey <u>all</u> axioms for fuzzy implications!

Implication	Valid Axioms
Kleene-Dienes	1 2 3 4 – 6 – 8 9
Reichenbach	1 2 3 4 – 6 – 8 9
Łukasiewicz	1 2 3 4 5 6 7 8 9 ←
Gödel	1 2 3 4 5 6 7 – –
Goguen	1 2 3 4 5 6 7 – 9
Zadeh	1 2 3 4 – – – – 9
Klir-Yuan	– 2 3 4 – – – – 9

characterization of fuzzy implication

Theorem:
$Imp: [0,1] \times [0,1] \to [0,1]$ satisfies axioms 1 - 9 for fuzzy implications for a certain fuzzy complement $c(\cdot)$ $\Leftrightarrow$

$\exists$ strictly monotone increasing, continuous function $f: [0,1] \to [0, \infty)$ with

- $f(0) = 0$
- $\forall a, b \in [0,1]: Imp(a, b) = f^{-1}(\min\{ f(1) – f(a) + f(b), f(1)\})$
- $\forall a \in [0,1]: c(a) = f^{-1}(f(1) – f(a))$

Proof: Smets & Magrez (1987), p. 337f. ■

examples: (in tutorial)

choosing an „appropriate" fuzzy implication ...

apt quotation: (Klir & Yuan 1995, p. 312)

„To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem."

guideline:
GMP, GMT, GHS should be compatible with MP, MT, HS
for fuzzy implication in calculations with relations:
$B(y) = \sup \{ t(A(x), Imp(A(x), B(y))) : x \in X \}$

example:
Gödel implication for t-norm = bounded difference